SHADES AND SHADOWS

COURSE IN DESCRIPTIVE GEOMETRY

FOR THE USE

OF

COLLEGES AND SCIENTIFIC SCHOOLS

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A COURSE

IN

SHADES AND SHADOWS.

FOR THE USE OF

COLLEGES AND SCIENTIFIC SCHOOLS.

BY

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SHADES AND SHADOWS.

THE GEOMETRICAL DELINEATION OF SHADES AND SHADOWS.

1. Light, whatever hypothesis we may adopt respecting its nature, is invariably propagated in right lines as long as it passes through the same medium; the right line along which the light holds its course is called a *ray of light*, and any collection of such rays, of definite thickness, is called a pencil.

In nature, light diverges in all possible directions from each luminous point, so that the pencils are all primarily diverging; but when the luminous origin is very distant, as in the case of a heavenly body, the rays in every pencil we consider are sensibly parallel. On account of the great distance between the earth and the sun, the rays may, without material error, be considered parallel; draughtsmen so consider them, thereby simplifying their constructions. In the following problems, the rays will be so assumed.

A ray of light from any point of a luminous body is represented by a right line.

A plane of rays is a plane passing through a ray.

SHADE OF A BODY.

2. Let Fig. 0, Plate I., B, be an opaque body illuminated by a pencil of solar rays whose direction is indicated by the arrow R. The surface of this cylindrical pencil touches B in a curve C. The portion of this cylinder from which the rays are excluded by B is called the indefinite shadow of B; and any object situated within this portion of the cylinder is in the shadow of B, or has the shadow of B cast upon it.

3. *Line of shade of a body.*

The curve C separates B into two parts: viz., that toward the source of light, called the illuminated part; and that opposite the source of light, called the shade. The rays are excluded from the shade by the body itself.

Any plane tangent to the cylindrical pencil must be tangent to B at some point of C (Art. 130, *Des. Geom.*); and, conversely, any plane tangent to B at a point of C will be tangent to this cylindrical pencil, and therefore contain a ray (Art. 111, *Des. Geom.*), and thus be a plane of rays. Hence we may always determine points on the line of shade of any opaque body by passing planes of rays tangent to the body, and finding their points of contact.

4. *Shadow of a body.*

If we suppose any surface, as a screen S, to be placed behind the body B, a part of S will have the shadow of B cast upon it at M. M is the portion of S from which light is excluded by B, or M is the shadow of B on S.

Line of shadow. The periphery C_1 of M, or the line which separates the illuminated portion of S from the shadow, is called the line of shadow. It is also the intersection of the surface of the cylindrical pencil with the surface on which the shadow is cast.

5. *Shadow of a curve.*

The pencil of rays passing through a curve forms a cylindrical surface; the intersection of this pencil with any other surface forms the shadow of the curve on that surface. Thus, in Fig. 0, C_1 is the shadow of C; i.e., *the line of shadow is always the shadow of the line of shade.*

6. The two cylindrical pencils passing through two tangent curves are tangent along the ray passing through the point of contact of the curves (Art. 131, *Des. Geom.*); hence the intersections of the pencils by any surface will be tangent at the point in which this ray pierces the surface; these intersections are the shadows of the curves, *hence the shadows of tangent curves are tangent to each other;* this is also true when one of the curves becomes a right line.

7. *The shadow of a plane curve on a plane.*

Designate the curve by C, and its shadow on the plane P by C_1; then, 1st, if the plane of C is a plane of rays, C_1 will be a right line; 2d, if the plane of C is parallel to P, then C_1 will be equal and parallel to C.

8. *The shadow of the circumference of a circle on a plane* is, in general, an ellipse[1] (Art. 140, *Des. Geom.*); and the shadows of any two diameters of the circle, taken at right angles to each other, are conjugate diameters of the ellipse of shadow.[2] Denote by A and B two such diameters of the circle, and by A_1 and B_1 their respective shadows; draw tangents T and U at the extremities of A: then as T, U, and B are parallel, their shadows T_1, U_1, and B_1 will also be parallel, and T_1 and U_1 will be tangent to the ellipse of shadow at the extremities of the diameter A_1: hence A_1 and B_1 are conjugate diameters of the ellipse of shadow.

9. *Shadow of a polyhedron.*

When the opaque body B is bounded by planes, the pencil of rays touching B has the form of a prism, whose exterior surface is made up of planes of rays not mathematically tangent to B. In this case, the lines of shade and shadow are broken lines.

10. NOTATION. As far as practicable, the shadows of points *a*, *b*, *c*, etc., will be denoted by a_1, b_1, c_1, etc.; those of lines A, B, C, etc., by A_1, B_1, C_1, etc.

The direction of a ray of light R will be given by its projections R^h and R^v, and the angle of R with the horizontal plane will be denoted by θ.

SHADOWS OF POINTS AND LINES.

11. When the opaque body is reduced to a point, the pencil becomes a single ray, and the shadow on a surface is the point in which this ray pierces the surface. Thus, Fig. 1, Plate I., a ray of light R, drawn through a given point *a*, pierces the coördinate plane V in the point a_1, which is therefore the shadow of *a* on V.

The point c_1, Fig. 2, in which a ray R, drawn through *c*, pierces the coördinate plane H, is the shadow of *c* on H.

12. The shadow of a right line on a plane is determined by finding the shadow cast by any two of its points on the plane: thus the shadow of the right line A, Fig. 3, on V, is found by constructing the shadows of any two of its points, as *a* and *b*. Drawing rays through *a* and *b*, we have their shadows a_1 and b_1; and the line $a_1 b_1$, or A_1 is the shadow required.

The two rays $a\,a_1$ and $b\,b_1$ determine a plane of rays.

The shadow of a right line A, on any surface S, may be found by passing a plane of rays through A, and finding its intersection with S.

CONVENTIONAL DIRECTION OF THE RAYS OF LIGHT.

13. In delineating the shadows of structures and machines, a conventional direction for the light has been adopted which presents great advantages, both in clearness of design and facility of construction.

We suppose the direction of the ray of light R, Fig. 4, to be the diagonal *oc* of a cube so placed as to have two of its faces parallel to the vertical, and two to the horizontal plane of projection. Consequently R^v and R^h make angles of 45° with the co_o; again, R', the projection of R on the profile plane oo_o, makes an angle of 45° with $o_o o^h$.

[1] EXCEPTIONS. 1°. When the plane is parallel to the curve. 2°. When the plane cuts a sub-contrary section from the pencil of rays (Art. 176, *Des. Geom.*).

[2] Two diameters of an ellipse are conjugate when one of them is parallel to tangents drawn through the vertices of the other.

If we call the side of the cube unity, *the particular value of θ for this case* is $\theta = oco^h$, whence

$$co^h = \sqrt{2}\,;\ \tan\theta = \frac{1}{\sqrt{2}} = \tfrac{1}{2}\sqrt{2} = 0.707,$$

$$oc = \sqrt{3}\,;\ \sin\theta = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3} = 0.577,$$

$$\cos 2\theta = \tfrac{1}{3};\ \cos\theta = \frac{\sqrt{2}}{\sqrt{3}} = \tfrac{1}{3}\sqrt{6} = 0.816,$$

$$\sec\theta = \frac{\sqrt{3}}{\sqrt{2}} = \tfrac{1}{2}\sqrt{6} = 1.225,$$

$$\theta = 35°15'\,52''\,;\ \operatorname{cosec}\theta = \sqrt{3} = 1.732.$$

The angle θ is easily constructed, Fig. 5, by assuming a point (o^h, o^v) on the ray (R^h, R^v), and revolving this ray around cx until it coincides with the vertical plane V, whence we have $ocG = \theta$.

14. *Advantages of assuming* R^h *and* R^v *at an angle of 45° with* GL.

To show how the construction may be shortened by assuming R^h and R^v at an angle of 45° with GL, we will suppose, Fig. 1, it was required to find the shadow of a; we will denote the given distance of the point a from the plane on which its shadow is to be cast (in this case, the vertical coördinate plane) by δ, i.e., $Ga^h = \delta$. Construction: Lay off $a^v o = \delta$, draw through o a line parallel to GL, and set off on it a distance $oa_1 = \delta$; a_1 is the shadow required.

15. In Fig. 2, the shadow of c on H is required. In this case the distance (δ) of c from the plane on which the shadow is cast is $c^v n$. Construction: Lay off $c^h x = \delta$, draw through x a line parallel to GL equal to δ, and its extremity c_1 is the shadow required.

16. PROBLEM 1. *To find the shadow on one of the coördinate planes, of a square, parallel to the other coördinate plane.*

Let, Fig. 6, $abcd$ be a square parallel to V: it is required to find its shadow on H.

1st method. Rays making any angle whatever. Draw through a, b, c, and d, rays, and their horizontal traces a_1, b_1, c_1, and d_1 will be the four angular points of the required shadow. We see that the lines ab and cd parallel to H have parallel shadows a_1b_1 and c_1d_1 on H.

17. *2d method.* R^h and R^v making angles of 45° with GL. Denote the given distance tc^v by δ, and the length of the side a^hb^h by l; make $a^hn = \delta$, $no = os = l$; erect at n, o, and s, three perpendiculars to GL, and lay off on the first the distance δ, on the second the distances δ and $\delta + l$, and on the third the distance $\delta + l$: the extremities of these perpendiculars will be the angular points sought; viz., c_1, d_1, a_1, b_1, whence the shadow is known.

18. PROBLEM 2. *To find the shadow on one of the coördinate planes of a right line perpendicular to either coördinate plane.*

Case 1st. Let A, Fig. 7, be a vertical right line: it is required to find its shadow on V. Assume any point (as n^v, A^h) on A, and draw through it a ray; this ray has its vertical trace at n_1, which is one point of the required shadow; since A is parallel to V, its shadow on V must be parallel to A and to A^v: therefore, drawing through n_1 a right line parallel to A^v, we have the required shadow A_1.

Abridged Construction. R^h and R^v making angles of 45° with GL. Let $tA^h = \delta$. Lay off $tx = \delta$, and erect a perpendicular A_1 to GL at x; A_1 is the shadow of the unlimited line A.

19. *Case 2d.* Let B, Fig. 8, be the right line perpendicular to V.

Construction. Assume any point of B (as o^h, B^v), and through it draw a ray; its trace on V is o_1, but B^v is the trace of the line itself on V: therefore the required shadow is B_1. In this case the shadow on V of an unlimited line B perpendicular to V is the unlimited vertical projection of the ray through B^v.

20. PROBLEM 3. *To find the shade of an upright prism, and its shadow on the vertical coördinate plane.*

Let the prism be given as in Fig. 9. If we pass vertical planes of rays through b^h and e^h, we perceive at once, that the faces horizontally projected in b^ha^h, a^hf^h, and f^he^h are illuminated, and those projected in b^hc^h, c^hd^h, and d^he^h are deprived of light and form the shade. The upper base is illuminated. The line of shade is composed of the edges (mb^v, b^h), bc, cd, (d^he^h, d^vc^v), and (c^vn, e^h). The only portion of the shade visible is the rectangle c^vnld^v.

The line of shadow is the shadow of these lines of shade: therefore, drawing rays through b, c, d, and (e^h, c^v), we have the points b_1, c_1, d_1, e_1, as the shadows of the angular points of the upper base; b_1, however, is invisible, as it is hidden by the prism itself. The shadow of the vertical edge through b is b_1o, invisible on the vertical plane: its shadow on the horizontal plane, as far as GL, is b^ho; the shadow of the vertical edge through (e^h, c^v) is e^hr on the plane H, and re_1 on V. Joining b_1c_1, c_1d_1, d_1e_1, we have $ob_1c_1d_1e_1r$ the portion on the plane V, and $re^hd^hc^hb^hor$ the portion on the plane H.

21. *Abridged construction.* Suppose, as is generally the case in architectural drawings, it were only required to find the shade and shadow on the vertical coördinate plane; R^h and R^v at 45° with GL.

The visible shade, as before, is c^vd^vln. The points o, p, q, and r may be found by laying off on GL the following distances: $mo = mb^h$; $np = nc^h$; $lq = ld^h$; and $nr = ne^h$; erecting perpendiculars to GL at o, p, q, r, and drawing the vertical projections of rays through b^v, c^v, d^v, their intersections with the respective perpendiculars give the points b_1, c_1, d_1, e_1, which, with e_1r, determine the shadow on the vertical plane.

22. PROBLEM 4. *To find the shade and shadow of an oblique cone.*

Let the cone be given as in Fig. 10. The shade is found by passing planes of rays tangent to the cone, and determining their elements of contact (Art. 3). For this purpose, draw a ray through the vertex o; o_1 is its horizontal trace. Drawing o_1n^h and o_1c^h tangent to the base of the cone, we have the horizontal traces of the two tangent planes of rays; the lines of contact are oc and on; hence noc is the line of shade. The portion of the shade visible on the horizontal projection is o^hlc^h; that visible on the vertical projection is $o^vl_0n^v$. The shadow on H is $c^ho_1n^h$.

23. PROBLEM 5. *To find the shade and shadow of a sphere.*

Let the sphere be represented as in Fig. 11, Plate II., and let the light have the direction indicated by R.

The surface of the cylindrical pencil of rays is tangent to the sphere, and has for its line of contact the circumference of a great circle, the line of shade, which is horizontally projected in an ellipse having c^h for its centre, and the horizontal line lc^hx drawn through c^h perpendicular to R^h for its transverse axis. The conjugate axis passes through c^h parallel to R^h; its length may be found as follows: Imagine a vertical plane of rays passed through the centre c of the sphere; oo_1 will be its horizontal trace; this plane will cut from the tangent pencil two rays tangent to the great circle cut from the sphere; this great circle will intersect the circle of shade[1] in a diameter having for its horizontal projection the required conjugate axis. The construction may be made as follows: Rotate this vertical plane of rays about its horizontal trace oo_1, until it coincides with H; c falls at $c_2(c^hc_2 = c_0c^v)$, and the circle o_2tn_2 is the revolved position of the intersection of the sphere with the plane of rays; the ray through the centre c, and the tangent rays, have the positions c_2c_1, n_2n_1, and o_2o_1, after rotation; the intersection of the two circles will be the line o_2n_2, drawn through c_2 and perpendicular to c_2c_1: making the counter revolution, o_2 and n_2 are found at o and n, whence on is the required conjugate axis.

The angle $c^hc_1c_2$ of the rays with H is equal to c_2n_2n; hence the required semi-conjugate axis is $c^hn = c_2n_2 \sin c_2n_2n$, i.e., $r \sin \theta$, denoting c_2n_2 by r.

The line of shadow of the sphere on H is cast by the circumference of shade: it is therefore an ellipse, with its centre at c_1, having for its conjugate axis l_1x_1 drawn through c_1 perpendicular to R^h and equal to lx. The transverse axis is the shadow of the diameter horizontally projected at on perpendicular to lx; its revolved position is o_2n_2; the rays through its extremities pierce H at o_1 and n_1. Hence o_1n_1 is the transverse axis required.

From the figure we have $n_1o_1 = n_2o_2 \operatorname{cosec} on_1n_2 = 2r \operatorname{cosec} \theta$.

The vertical projection of the line of shade. ef, the vertical projection of the diameter ef parallel to V, is the required transverse axis. To find the conjugate axis: The ray and the circle of shade are perpendicular to each other, and therefore make complementary angles with V. If we denote the angle of R with V by ϕ, found as in Fig. a, we shall have $90° - \phi$ as the angle of the circle of shade with V; hence (Art. 169, *Des. Geom.*) the required semi-conjugate axis c^vs is radius $\times \sin \phi$. In Fig. a, laying off $oa^v = oa^h = c^vf$, Fig. 11, we have cs as the required semi-conjugate axis which may be laid off in Fig. 11, and the ellipse of shade readily constructed.

[1] For brevity we will call the circle, having the line of shade for its circumference, the *circle of shade*.

This method might have been used to determine the horizontal projection of the line of shade.

24. *Abridged construction.* R^h and R^v at 45° with GL; denote the diameter of the sphere by d: then (Fig. 11) $qs = on = d \sin\theta = 0.577d$; $o_1n_1 = d \operatorname{cosec}\theta = 1.732d$ (Art. 13).

25. THEOREM 1. *When we cut off by a plane, and remove a portion of a surface of the second order, such as a cylinder, a cone, an hyperboloid, an ellipsoid, etc., the shadow of the section cast upon the interior surface so exposed is a plane curve, and consequently one of the second order.*[1]

26. PROBLEM 6. *To find the shadow of the edge of a hollow hemispherical shell upon its interior surface.*

Let the hemispherical shell be represented as in Fig. 13; and let the direction of the light be indicated by the arrows as parallel to V, and making an angle θ with H. This problem is taken as an illustration of the foregoing theorem. The two surfaces of the second degree are the hemisphere, and the half-cylinder of rays entering the hemisphere in the semicircle, whose projections are a^vc^v, ed^hf. The entering curve being thus a plane curve, and one of the second order, the exit curve must also be a plane curve of the second order. The only curve which a plane can cut from the sphere is a circle: the curve of shade in space is, therefore, a semi-circumference; it is horizontally projected in a semi-ellipse, of which ef is the transverse axis.[1] To find the semi-conjugate axis c^hs^h: Pass a vertical plane of rays through c, it will cut from the illuminating pencil the ray as, whence c^hs^h is the semi-conjugate axis of the horizontal projection of this semicircle. Join c^v and s^v: then the angle $s^vc^vx = 2\theta$, and $c^vx = c^vs^v \cos 2\theta$. If $\theta = 35° \; 15' \; 52''$, then, Art. 13, $\cos 2\theta = \frac{1}{3}$; hence $c^vx = \frac{1}{3}c^vs = c^hs^h$.

27. PROBLEM 7. *To construct the shadow of a niche upon its interior surface.*

The niche (Fig. 12) is an upright hollow semi-cylinder, projected vertically in the rectangle db^v and horizontally in the semicircle $a^hs_1{}^hb^h$, terminated by a quadrant of a sphere vertically projected in the semicircle ds^vf, and horizontally on the base of the semi-cylinder. R^v and R^h are assumed at an angle of 45° with GL. The line of shadow is divided into three portions: 1st, fx cast by the shadow of the arc A upon the spherical surface; 2d, $xd_1{}^v$ cast by A on the cylindrical surface; 3d, $n_1d_1{}^v$ cast by the element (a^vd, a^h) on the cylindrical surface. fx (by Art. 26) is the arc of an ellipse extending from f, the point of contact of the tangent ray, to x, having for its semi-axes zf, the radius, and $zo = \frac{1}{3}zf$; any point of $xd_1{}^v$ as $s_1{}^v$ may be found by passing a ray $(s^vs_1{}^v, s^hs_1{}^h)$ through s, and finding its trace $s_1{}^v$ on the surface of the cylinder; $d_1{}^vn_1$ is the shadow of dn, a portion of (da^v, a^h), upon the interior surface of the cylinder; n is found by drawing n_1n parallel to R^v.

SHADOWS OF CIRCLES.

28. Since in architectural drawings it is often required to find the shadows of circles in various

[1] This theorem is usually expressed thus, the surfaces being understood to mean those of the second order:—

If one surface enters [intersects] another in a curve of the second order, it will leave it [intersect it again] in a curve of the same order. This is a very well-known theorem of analytic geometry. The demonstration here given is due to M. Binnet.

Let us imagine two surfaces of the second degree; suppose XY to be the plane of the curve in which one surface enters the other.

The two equations of the surfaces may be written in their most general form, thus:—

$$ax^2 + by^2 + cz^2 + lxy + mxz + nyz + px + qy + rz + d = 0 \quad (1)$$

$$a'x^2+b'y^2+c'z^2+l'xy+m'xz+n'yz+p'x+q'y+r'z+d'=0 \quad (2)$$

The curve of intersection being contained in the plane XY.

If we put $z = 0$ in (1) and (2), we shall have

$$ax^2 + by^2 + lxy + px + qy + d = 0 \quad (3)$$

$$a'x^2 + b'y^2 + l'xy + p'x + q'y + d' = 0 \quad (4)$$

each of which represents the known curve of intersection of the two surfaces; (3) and (4) are identical, and consequently the co-efficients of the corresponding terms must be equal, or cannot differ except by a constant factor λ; hence,

$$a = \lambda a'; \; b = \lambda b'; \; l = \lambda l'; \; p = \lambda p'; \; q = \lambda q'; \; d = \lambda d'; \quad (5)$$

if we now multiply (2) by λ, and subtract it from (1), taking account of equations (5) after performing the subtractions and substitutions, we have

$$(c - \lambda c')z^2 + (m - \lambda m')xz + (n - \lambda n')yz + (r - \lambda r')z = 0 \quad (6)$$

which must be satisfied for all values of x, y, and z, common to the two surfaces. Since z is a common factor, (6) is satisfied by placing $z = 0$, or

$$(c - \lambda c')z + (m - \lambda m')x + (n - \lambda n')y + (r - \lambda r') = 0 \quad (7)$$

$z = 0$, belongs to the plane XY, in which the known curve of intersection lies; (7) is the equation of a plane, which, by its combination with (1) or (2), will give another curve common to both surfaces, and this curve must, of course, be one of the second order.

[1] As the semi-circumference cut from the hemisphere has (cf, e^v) for its diameter, it must be half of the circumference of a great circle.

positions, we give here two examples: R^v and R^h at 45° with GL.

Problem 8. *To find the shadow cast by a given circle parallel to one coördinate plane on the other coördinate plane.*

Let the circle be represented as in Fig. 14, vertically projected in $k^v g e^v$, and horizontally in $k^h e^h$. Imagine a square lr circumscribed about the circle, mr being horizontal. The shadow of this square on the coördinate plane H (Art. 16) is found by drawing $k^h m_1$ and $e^h r_1$ parallel to R^h, $e^h m_1$ perpendicular and equal to $k^h e^h$, and $m_1 r_1$ parallel and equal to $k^h e^h$; the middle points g, e, f, k have g_1, e_1, f_1, k_1 for their shadows. The shadow of the inscribed circle is an ellipse tangent to the parallelogram $k^h r_1$ at the points g_1, e_1, f_1, k_1, and having $g_1 f_1$ and $e_1 k_1$ for conjugate diameters (Art. 8).

To find the position and magnitude of the axes of the ellipse of shadow. Construction: Lay off $mn =$ to the radius of the given circle, join n with c, and produce nc to o; then no is equal to the required transverse axis, nx the conjugate, and nck^v double the angle of the transverse axis with the horizontal conjugate diameter; i.e., $no = s_1 s_2$; $nx = z_1 z_2$; and $s_2 c_1 e_1 = \frac{1}{2} nck^v$.[1]

[1] Verification of the construction given in Fig. 14. We assume the three equations of Analytic Geometry relating to conjugate diameters, as follows:—

$$a'b' \sin(a' - a) = ab \quad (1) \qquad a'b' \sin(a' - a) = ab = r^2$$
$$a'^2 + b'^2 = a^2 + b^2 \quad (2) \qquad a'^2 + b'^2 = 3r^2$$
$$\tan a' \tan a = -\frac{b^2}{a^2} \quad (3) \qquad 2ab = 2r^2$$

in which

$r =$ radius of the given circle

$a' = r\sqrt{2}$; $b' = r$

$a' - a = 45°$; $\sin 45° = \frac{1}{2}\sqrt{2}$

$$a + b = r\sqrt{5}$$
$$a - b = r$$
$$a^2 - b^2 = r^2\sqrt{5}$$

$$a = \frac{r}{2}(\sqrt{5} + 1);\; b = \frac{r}{2}(\sqrt{5} - 1);\; -\frac{b^2}{a^2} = -\frac{3 - \sqrt{5}}{3 + \sqrt{5}} = \frac{3\sqrt{5} - 7}{2}$$

substituting these values in (3), we have

$$\tan a \tan(45 + a) = \frac{\tan a(1 + \tan a)}{1 - \tan a} = \frac{3\sqrt{5} - 7}{2}$$

for brevity let $y = \tan a$, then

$$\frac{y + y^2}{1 - y} = \frac{3\sqrt{5} - 7}{2}, \text{ or } y^2 + \frac{3\sqrt{5} - 5}{2} y = \frac{3\sqrt{5} - 7}{2}$$

whence $y = \tan a = 2 - \sqrt{5}$, or $\frac{1 - \sqrt{5}}{2}$; whence $\tan 2a = -2$ or $-\frac{1}{2}$; $2a = 116°34'$ or $-63°26'$; whence $a = 58°17'$ or $-31°43'$; whence $s_2 c_1 e_1 = a = -31°43'$; $s_2 c_1 g_1 = a' = 45 + a = 45° - 31°43' = 13°17'$; or $z_2 c_1 e_1 = 58°17' = a$ and $a + 45 = a' = 103°17' = z_2 c_1 g_1$. Using the other value, we obtain $\tan 2a = -\frac{1}{2}$; we have $2a = 153°26'$ or $-26°34'$; hence we have as another value, $a = 76°43'$ or $-13°17'$; whence $a' = 121°43'$ or $31°43'$. In Fig. 15, $s_2 c_1 g_1 = -13°17'$, and $s_2 c_1 e_1 = 31°43'$; also, $z_2 c_1 g_1 = 76°43'$, and $z_2 c_1 e_1 = 121°43'$.

29. Problem 9. *To find the shadow of a circle situated in the profile plane.*

Let the circle (Fig. 15) be given by its centre c, and the length of its diameter $fg = ek$; R^h and R^v being taken at 45° with GL. Imagine a square, circumscribing the circle, two of its sides parallel to the plane H, and the other two perpendicular to it. The shadow of the square on H is formed of the two lines km and er drawn through k and e parallel to R^h (Art. 16), and two others ek and mr perpendicular to GL and separated by an interval $em = fg$. The shadow of the circle is tangent at the points c^h, c_1, g_1, k_1; and $c^h g_1$ and $e_1 k_1$ are its conjugate diameters. The axes $s_1 s_2$, $z_1 z_2$, are found by Fig. 14.

30. *Construction of the axes of an ellipse from two given conjugate diameters.* Let ce and cn (Fig. 16) be the conjugate diameters, making a given angle ω with each other. Construction: From the extremity n, of one diameter, drop a perpendicular nf on the other produced; from n set off distances $nl = nr = ce$; join c with l and r, and draw through n a parallel nd to cr, intersecting cl in o; make $ox = od = ol$, and join c with d and x; make $ca = nd$, and $cb = nx$, then cb and ca will be the new axes both in magnitude and position.[1]

[1] The complete verification of this construction is made by showing that the new axes in magnitude and position satisfy the usual equations of Analytic Geometry relative to conjugate diameters; viz.,—

$$a'b' \sin(a' - a) = ab \quad (1) \qquad dn = a$$
$$a^2 + b^2 = a'^2 + b'^2 \quad (2) \qquad nx = b$$
$$\tan a \tan a' = -\frac{b^2}{a^2} \quad (3) \qquad a' - a = \omega$$

angle $nrc = r$ $\qquad cn = a'$

$ncr = \phi$ $\qquad ce = b'$

$nca = \gamma = dcn - 90°$ $\qquad ecn = \omega$

Drop from d a perpendicular dz on nc produced; then $cdz = \gamma = nca$. With o as a centre, and a radius cl describing a circumference, we have from the two secants nd and nf, $ab = a'b' \sin \omega$ (1).

In the triangle cfr, $cr = \sqrt{cf^2 + fr^2}$
$$= \sqrt{a'^2 \cos^2\omega + (a' \sin \omega + b')^2} = a + b.$$

In the triangle clr, since $ln = lr = b'$, we have $\overline{cr}^2 + \overline{cl}^2 = 2\overline{cn}^2 + 2\overline{nl}^2$, or $(a + b)^2 + (a - b)^2 = 2a'^2 + 2b'^2$; whence $a^2 + b^2 = a'^2 + b'^2$ (2).

$$\sin r = \frac{a' \cos \omega}{a + b};\quad \sin \phi = \frac{b' \sin r}{a'} = \frac{b' \cos \omega}{a + b};$$
$$\cos^2\phi = 1 - \sin^2\phi = \frac{(a + b)^2 - b'^2 \cos^2\omega}{(a + b)^2} = \frac{(a' + b' \sin \omega)^2}{(a + b)^2};$$
$$cz = a \cos \phi - a' = \frac{ab' \sin \omega - a'b}{a + b};$$
$$zd = a \sin \phi = \frac{ab' \cos \omega}{a + b};$$

31. PROBLEM 10. *To construct the shade of a right cylinder with a circular base, and the shadow of the upper circle on the interior surface.*

Let the cylinder be given as in Fig. 17. 1st, To find the shade. The tangent planes of rays touch the cylinder along the vertical elements through r and t, thus giving r^vf^vbp as the shade visible on V.

The shadow on the interior surface is cast by the semicircle ter. The line of shadow is (Art. 25) a plane curve, an ellipse, having tr for a diameter and c for its centre. Other conjugate diameters of this ellipse may be found (Art. 8) by selecting pairs of radii of the semicircle ter at right angles to each other and finding their shadows; besides ct we will take c (l, t^v); also ce and c (y, c^v). Through the extremities of these radii t, e (l, t^v), (y, c^v), pass rays and vertical planes of rays; these planes cut from the cylindrical surface elements horizontally projected at t^h, u, z, and f^h; and the rays intersect these elements in the points e_1, l_1, and y_1, the shadows of the assumed extremities; hence joining c with e_1, l_1, and y_1, we have two pairs of semi-conjugate diameters vertically projected in c^vt^v, c^vl_1, and c^ve_1, c^vy_1. Since c^vt^v and c^vl_1 are one pair of semi-conjugate diameters, the line gl_1 parallel to c^vt^v must be tangent to the vertical projection of the ellipse of shadow. For the same reason t^vg parallel to c^vl_1 and a line parallel to it at r^v are also tangents at t^v and r^v; f^vy_1 parallel to c^ve_1 is also tangent at y_1: we have, therefore, five points of the vertical projection of the line of shade, viz., t^v, c_1, l_1, y_1, r^v, and also the directions of the five corresponding tangents. Hence we have the following *simple construction for the ellipse of shadow.* Determine as above the shadows y_1 and l_1 of the points (y, c^v) and (l, t^v), draw e_1y_1 and gl_1 parallel to GL; join gt^v (g being on the axis); then we have the five points t^v, e_1, l_1, y_1 and r^v, and the five corresponding tangents; viz., t^vg, e^ve_1, gl_1, f^vb, and a parallel to gt^v through r^v, which are entirely sufficient to determine the shadow.

The shadow on any particular element, e.g., that on (f^vb, f^h), is found by passing a plane of rays through this element, and finding its intersection (c^v, y) with the semi-circumference ter, whence, drawing a ray through (c^v, y), we have the required shadow y_1.

The lowest point of the shadow. Since the distance c^ve_1 of any point e_1 of the shadow below the upper base is proportional to the distance e^hu between the element casting, and that receiving the shadow, it follows that this vertical distance will be greatest when the distance between these elements is a maximum, i.e., when the plane of rays passes through the axis.

32. *Abridged construction.* R^v *and* R^h *at* 45° *with* GL. Denote the radius of the base of the cylinder by ρ; then $bp = \rho$ versin 45° $= \frac{3}{10}\rho$[1]; $c^ve_1 = \rho$; $c^vy_1 = \rho\sqrt{2}$; the angle $y_1c^ve_1 = 45°$; hence the axes of the ellipse of shadow may be constructed as in Art. 28: therefore making $e^vn = 2\rho$, and with c^v as a centre describing the semi-circumference, xe^vo, no, and nx are the lengths of the axes of the ellipse of shadow, and the angle $nc^ve^v = 2\gamma$.

33. *Simplified construction.* Lay off $c^vt^v = c^vr^v = \frac{7}{10}\rho$, and draw the elements r^vp and c^vs; set off $c^ve_1 = c^ve^v$, and $c^vg = 2c^vt^v$; draw e_1y_1 and gl_1 parallel to GL; their intersections with f^vb and r^vp determine y_1 and l_1. We then have the five points t^v, e_1, l_1, y_1 and r^v, and their corresponding tangents t^vg, e^ve_1, gl_1, f^vy_1, and a right line through r^v parallel to t^vg.

34. PROBLEM 11. *To find the shadow of a rectangular abacus on a cylindrical column, and on the vertical coördinate plane.*

$$\frac{cz}{zd} = \tan\gamma = \frac{ab'\sin\omega - a'b}{ab'\cos\omega}; \qquad (4)$$

$-\gamma = a$; $\omega - \gamma = a'$. Substituting in (1), we have

$$-\tan\gamma\tan(\omega-\gamma) = -\frac{b^2}{a^2};$$

hence

$$\frac{\tan\gamma\tan\omega - \tan^2\gamma}{1 + \tan\omega\tan\gamma} = \frac{b^2}{a^2}.$$

Substituting, we have for the numerator, —

$$\frac{(ab'\sin^2\omega - a'b\sin\omega)ab'}{D^2} - \frac{a^2b'^2\sin^2\omega - 2a^2b^2 + a'^2b^2}{D^2} = \frac{b^2(a^2 - a'^2)}{D^2}$$

D denoting the denominator $ab'\cos\omega$, and remembering that $a'b'\sin\omega = ab$. For the denominator

$$\frac{a^2b'^2\cos^2\omega + a^2b'^2\sin^2\omega - aba'b'\sin\omega}{D^2} = \frac{a^2(b'^2 - b^2)}{D^2};$$

since $a^2 + b^2 = a'^2 + b'^2$, we have $\frac{a^2 - a'^2}{b'^2 - b^2} = 1$;

hence $\tan\gamma\tan(\omega - \gamma) = \frac{b^2}{a^2}$; or $\tan a' \tan a = -\frac{b^2}{a^2}$, which is (3). Substituting in (4) the values of a', b', a, b, and ω, given in the note to Art. 28, we have $\tan\gamma = 2 - \sqrt{5}$, or $\tan a = \sqrt{5} - 2$, which is the same as that given in Art. 28.

[1] If the diameter is $0^m.1$, then $\frac{3}{10}$ of $0^m.05$ is accurate to within $0^m.0005$.

Let (Fig. 19) $a^v e^v$ and $a^h d$ be the projections of the abacus, and $p^v y$ and $p^h x r^h$ those of the semi-column.

1°. *The shadow of the edge* $a^h i, a^v$ is obtained by passing a plane of rays through it: this plane will be perpendicular to the vertical plane, and have $a^v a_1^v$ for its vertical trace. The ray through a pierces the column at a_1, which is the end of this shadow; $a^v l$ is the shadow on V; $(la_1^v, p^h a_1^h)$ is the shadow on the cylinder; the triangle $a^v lq$ is the shadow cast on V by a portion of the lower surface of the abacus.

2°. *The shadow cast by ab on the cylinder* is the curve $a_1^v ny$; that of ab on V is $u_1 b_1$, obtained by drawing rays through a, u, and b, and finding the points in which they pierce the cylinder and V. The curve is, in general, an ellipse. $u^h t$, the trace of the plane of rays tangent to the column, determines yz, the line of shade of the column, and the shadow $u_1 x_1$ of a portion of (zy, x). The line $(u^v y, u^h x)$ is in the tangent plane, and in the plane of rays; therefore it is their intersection, and tangent to the curve of shade at y; y is the point where the line of shadow disappears in the shade of the cylinder. The shadow of $(b^v e^v, b^h)$ is $b_1 e_1$; that of (e^v, db^h) is $e^v e_1$.

35. *Abridged construction.* R^v and R^h at 45° with GL.

Pass a plane of rays through the axis of the column; $c^h s^h$ is its horizontal trace; this plane intersects the edge ab in s, and $s^v o$ is the vertical projection of the ray through s. Denote the length of the radius of the column by ρ.

With o as a centre and ρ as a radius, describe an arc $s_1^v ny$; this arc will be the vertical projection of the shadow of ab on the column[1]; take $zr^v = \frac{3}{10}\rho$, and $zx_1 = c^v z$; erecting a perpendicular at x_1, and drawing yu at 45°, we have u_1. Drawing rays through b and e, we complete the outline by drawing $u_1 b_1$ parallel, and $b_1 e_1$ perpendicular, respectively, to GL.

36. *Shadow of a point on a cylinder.* R^v and R^h at 45° with GL. We have seen (Art. 35) that the shadow of a (Fig. 19) on the column was a_1, which might be found as follows: Draw through a^v a line parallel to GL, meeting the axis of the cylinder in b'. Set off $b'o = a^h i$, the distance of a in front of the axis: o is the centre of $s_1^v ny$, and the radius is that of the cylinder. Drawing through a^v a line parallel to R^v, we have the shadow a_1 of a upon the cylinder, without drawing any line on H. If the cylinder were hollow, and the shadow were required upon the interior surface, $b'o$ would be laid off above $a^v b'$, instead of below it.

37. PROBLEM 12. *To find the shade of a cylindrical column and of its cylindrical abacus; also the shadow of the abacus, both on the column and on the vertical coördinate plane.*

Let the abacus and column be represented as in Fig. 20, the projections of the abacus being $y^v n^v$ and $y^h e^h n^h$; those of the column, kt and $\alpha^h \delta^h l^h$.

Let the column be intersected by vertical planes of rays cutting the lower base of the abacus in points a, b, d, etc., and the surface of the column in elements horizontally projected in α^h, β^h, δ^h, etc.; drawing rays through a, b, d, etc., of the abacus, we have their corresponding shadows α, β, δ, etc.

The lines of shade and shadow of the column are determined by its tangent planes of rays: these lines are $(\epsilon^v \epsilon', \epsilon^h)$ and $(\epsilon_1 o^v, o^h)$: $(w^v w_2^v, w^h)$ is the line of shade of the abacus. The line of shadow of the abacus on V, cast by the arc $(y^v w_2^v, y^h u^h)$, is a portion of an ellipse beginning at y and extending to α: here it is intercepted by the column and its shadow as far as ϵ_1, where it begins again, and extends to w_2; $w_1 w_2$ is the shadow of $(w^v w_2^v, w^h)$; $w_1 n^v$ is the shadow of the upper edge of the abacus wn.

BRILLIANT POINTS.

38. When a pencil of rays falls on a polished surface S, one ray, at least, is usually reflected to

[1] Let o be the origin of a system of rectangular coördinate planes, ob' the axis of Z; the plane ZX the vertical plane having ik for its horizontal trace. [If we refer to Fig. 4, Plate I., we see that a plane of rays through ot makes an angle of 45° with H.] The equation of any plane is $z = cx + dy + g$ (1); the angle ϕ, which this plane makes with the plane XY, is $\cos\phi = \dfrac{1}{\sqrt{1 + c^2 + d^2}}$ (2). The equations of the cylinder are $x^2 + y^2 = \rho^2$, z indeterminate (3). Let $ob' = c^h f = l$; $\phi = 45°$; $\cos\phi = \sqrt{\frac{1}{2}}$. The equation of the plane of rays through ab may be found as follows: Since ab is parallel to the axis of X, $c = o$; since it makes an angle of 45° with the plane XY, $\cos^2\phi = \frac{1}{2}$; hence $\frac{1}{2} = \dfrac{1}{1 + d^2}$ or $d = 1$. When $x = l$, $z = l$; hence $l = l + g$; $\therefore$ $g = o$; and the equation of the plane of rays is $z = y$, x indeterminate (4). Substituting this value of y in (3), we obtain, as the equation of the vertical projection of the intersection of the cylinder by the plane of rays, $x^2 + z^2 = \rho^2$; the equation of a circumference with o as a centre.

the eye of the observer, who thus sees on S one or more *brilliant points.*

General Construction. Let us now suppose S to be represented on the vertical plane V. Let R be the incident ray; Z, perpendicular to V, the reflected ray passing through the point of sight; x the brilliant point, and N a normal to S at x. Then, according to a well-known principle of optics, R, N, and Z must be in the same plane; also, R and Z make equal angles with N on opposite sides.

The directions of R and Z being known, that of N is found by bisecting the angle of R with Z; the brilliant point x will be the point of contact of a plane tangent to S and perpendicular to N.

39. PROBLEM 13. *To find the brilliant point on a spherical surface.*

This example is taken to illustrate the general method given in Art. 38.

Fig. 26 represents a quadrant of a sphere, the centre c being in GL; R a ray of light to the centre c; R′ is the position of the ray revolved to coincide with H; cn is the bisecting normal in its revolved position; and d the real position of the required brilliant point.

40. PROBLEM 14. *To find the brilliant point on a surface of revolution.*

Let the surface be given as in Fig. 21, to find the brilliant point on its vertical projection. Through any point of GL, as o, draw a ray ao; oz perpendicular to GL is the required direction of the reflected ray, to pass through the point of sight. Bisect the angle of ao with oz (Art. 40, *Des. Geom.*). [Rotate ao about oz till it coincides with H; it takes the position $a'o$; ob' bisects $a'oz$; making the counter revolution, it takes the position ob.] Any plane perpendicular to ob, such as the one having for its traces on and op, is parallel to the plane tangent to the surface at its brilliant point; denote this unknown plane by X; then, since the meridian plane M through the brilliant point is perpendicular to X (Art. 133, *Des. Geom.*), it must be perpendicular to the plane nop parallel to X; therefore, ce perpendicular to op is the horizontal trace of M, and cx its vertical trace; again, X and nop being parallel, their intersections with M must be parallel; but the intersection of M with nop is the line having n and r^h for its traces, and nr' as its revolved position when M is rotated about cx so as to coincide with V. Draw $d''g$ tangent to U^v and parallel to nr'. d'' is the revolved position of the point of contact; making the counter revolution, (d'', d') is found at (d^h, d^v), $d''d^v$ being parallel to GL. Hence d is the brilliant point required.

This construction gives the brilliant point only on the vertical projection of the surface: a similar construction will give the brilliant point on the horizontal projection, these points being entirely distinct from each other.

Although two tangents can be drawn to U^v parallel to nr', gd'' only determines the real brilliant point.

When R^v and R^h are at 45° with GL, the angle $b^hoz = ecy = 20° \, 6' \, 14''$[1].

SHADOWS OF POINTS ON CURVED SURFACES.

41. We have given (Art. 156, *Des. Geom.*) a general method of finding the point in which a right line pierces a surface: we shall now apply it to the cone and sphere.

PROBLEM 15. *To find the shadow of a given point on the surface of a cone.*

Let a, Fig. 10, be the given point; pass a plane of rays through a so as to cut the simplest line from the cone: for this purpose draw two rays through a and o; a_1 and o_1 are their horizontal traces, and o_1s the trace of the auxiliary plane: this plane cuts from the cone the element os; which intersects the ray through a at the point α, the required shadow.

42. PROBLEM 16. *To find the shadow of a given point on the surface of a sphere.*

Let the sphere be given as in Fig. 28, its centre in GL, and let a be the point. Pass through a a plane of rays perpendicular to V, intersecting the sphere in a circle having xz for its diameter; rotating this plane about its vertical trace a^vz until it coincides with V, $xa_1'z$ is the revolved position of the circle, and $a'z$ that of the ray. $a^va' = a_oa^h$; a_1' is the revolved position of the shadow of a upon the sphere; making the counter revolution, a_1' returns to a_1, which is the

[1] Denote the angle $a'oz$ by ϕ, then $\tan\phi = \frac{oa^v}{oa^v \cos 45°} = \sqrt{2}$, $\tan\frac{\phi}{2} = \text{cosec}\,\phi - \cot\phi = \sqrt{\tfrac{3}{2}} - \sqrt{\tfrac{1}{2}}$; $\tan b^hoz = \frac{b^vo \cos 45°}{bvo \tan ob'b^h} = \frac{1}{\sqrt{2}}\tan\frac{\phi}{2} = \frac{1}{\sqrt{2}}\frac{\sqrt{3}-1}{\sqrt{2}} = \frac{\sqrt{3}-1}{2} = 0.3660254 = \tan$ of $20° \, 6' \, 14''$. But $b^hoz = r^hoc = 20° \, 6' \, 14''$.

shadow of a. If the distance $a_o a^h = \delta = a^v a'$ be given, no construction on the horizontal coördinate plane is required.

NATURE OF THE LINE OF SHADE FOR SIMPLE SURFACES.

43. We have already seen that the line of shade of a sphere is the circumference of a circle, that of a polyhedron is a broken line, while that of a cylinder or cone is made up of two elements of the surface.

PROBLEM 17. *To find the line of shade of a surface of the second order.*

Let S denote such a surface having a centre, and for the moment suppose the source of light reduced to a point; then the line of shade is a plane curve.

Proof. Let s, Fig. 35, be the luminous point, and c the centre of the surface; draw through s and c a secant plane; it cuts from the surface a curve mdt. We obtain two points, m and t, of the line of shade, by drawing the tangents sm and st. Draw the chord mt, and join s and c; sc intersects the curve at x and the chord at n. Then we know that the chord is parallel to the tangent xy of the curve, and also that $cn \times cs = \overline{cx}^2$.[1]

We conclude from this, that, whatever secant plane is drawn through cs, the point n is always the same. Also, *for each* secant plane, we draw through n a right line parallel to a tangent at x; but all these tangents at x belong to the tangent plane to the curved surface at x, and the locus of all the parallels to this plane drawn through n is a plane through n parallel to the tangent plane at x, and the intersection of S by this plane is the line of shade, which is a curve of the second order.

When s is at an infinite distance, as in the case of parallel rays, the plane of the curve of shade passes through the centre; and if the surface of the second order has no centre, the plane of the curve of shade is parallel to the axis.

[1] The second portion becomes evident when we see that uy must be parallel to the diameter b' conjugate to cx, which we will denote by a'; then denoting cn by x', and nm by y', we have the equation of the tangent ms, $a'^2yy' \pm b'^2xx' = \pm a'^2b'^2$; as s is on the axis of x, for the point s we have $y = o$ and $\therefore xx' = a^2$ or $cn \times cs = \overline{cx}^2$, Q. E. D.

If mdt were a parabola, then any diameter $\overline{sc}$ must bisect a system of chords parallel to $\overline{uy}$; also $\overline{sn} = 2\overline{xn}$.

44. *There are three general methods of finding the line of shade of a surface.*

1st, *The method of secant planes.* This consists in intersecting the surface S by a series of planes of rays cutting from S curves C′, C″, C‴, etc., and from the surface of the illuminating pencil of rays tangents to these curves: the points of contact are points of the required line of shade.

The method of tangent planes already used may be cited in this connection.

2d, *The method of circumscribed surfaces.* For example, let, Fig. 29, a cone and cylinder circumscribe a sphere having the circumferences bc and ef as the respective lines of contact. Then the plane of rays tangent to the cone along an must be tangent to the sphere at n, and, therefore, n is a point of the line of shade of the sphere: for a similar reason, s is also a point of the same line of shade.

Remark. Fig. 29 illustrates the fact that when two surfaces are tangent, the lines of shade are not therefore tangent, as an and st make acute angles with $rsnk$.

3d, *The method of oblique projections.* Let, Fig. C, Pl. V., C and D be two curves in space. It is required to find the point on C which casts its shadow on D. For this purpose we find, on an auxiliary plane P, the shadows C_1 and D_1 of the two given curves, which are oblique projections of C and D. The point of intersection x_1 of C_1 and D_1 is the trace on P of the ray which meets C and D; its point of meeting x on C and x_2 on D are the points required.

The principle may be thus enunciated: To determine the shadow cast by one curve upon another, *Find the oblique projections of the given curves upon the same plane, and the points of intersection of these projections are the traces of the projecting lines which intersect both of the given curves.*

45. *To determine the points of the line of shade situated upon the apparent contour.*

The apparent contour of a surface is the base of a cylinder circumscribing the surface and perpendicular to the plane of projection. Any curve, as the line of shade, traced around the surface, is tangent, in projection, to the apparent contour, the points of contact being the traces on the plane of projection of the elements of shade of the projecting cylinder: these traces are the points of contact of the apparent contour with tangents drawn parallel to the projection of the light.

The foregoing remarks will now be applied in the solution of the following: —

46. PROBLEM 18. *To find the line of shade of a torus.*

Let the torus be represented as in Fig. 22, R^v and R^h at 45° with GL.

Points on the contour lines X^v *and* Y^h are found by drawing tangents to these lines parallel respectively to R^v and R^h; thus determining the points α^v, ϵ^v, β^h, η^h, in which the projections of the line of shade are tangent to the contour lines of the torus: α^h, ϵ^h, β^v, and η^v are found on X^h and Y^v respectively.

The highest and lowest points are in a vertical plane of rays P, drawn through the centre; for the illumination is the same on each side of P, which therefore divides the line of shade, as well as the surface, symmetrically. Again, P cuts from the torus a meridian, and, from the illuminating pencil, two rays tangent to it: rotating this meridian about the axis until its horizontal trace HP becomes parallel to GL, and drawing tangents making an angle θ with GL, we have the revolved position x and z of the highest and lowest points; making the counter revolution, x and z take the positions ν and δ.

Points on the profile meridian. On account of the symmetry of position of the principal and the profile meridians with respect to the direction of the light, each point of shade on the first corresponds to a point of the second upon the same parallel. Hence the arcs $\alpha\lambda$ and $\epsilon\gamma$ determine the required points λ and γ.

47. *Abridged method for finding the visible line of shade on the vertical plane of projection*, R^v and R^h still being at 45° with GL.

Let the vertical projection of the torus be given, as in Fig. 23. Tangents to the vertical contour line parallel to R^v determine the points α and ϵ; $\epsilon\gamma$, drawn parallel to GL, determines γ at its intersection with the axis on.

In Fig. 22, $ob = o\beta^h \cos 45° = 0.7\overline{o\beta}_h$. Hence $o\beta$, Fig. 23, $= oc = \overline{om} \cos 45° = .7\overline{om}$. Draw to the contour a tangent zn, making an angle θ with om, and produce it to the axis at n. z is the revolved position of the lowest point; making the counter revolution, n remains in the axis, and nz takes the direction $n\delta$, parallel to R^v; drawing $z\delta$ parallel to om, we have δ as the intersection of $n\delta$ and $z\delta$. We have then five points and three tangents to $\alpha\beta\gamma\delta\epsilon$, which are abundantly sufficient for its construction in practice.

48. PROBLEM 19. *To find the line of shade of a surface of revolution: general method.* We have shown, in the previous problem, how to find the highest and lowest points of the line of shade, and also those on the contour lines. The object in the present problem is to show how to find any point whatever of the line of shade. Let the surface of revolution be given as in Fig. 27 (o^vz, o^h), a vertical axis, $b^vn^ve^v$ the vertical projection of the meridian curve, e^hf^h its horizontal projection. It is required to find any point of the line of shade, as that situated on the parallel np.

49. *First method, by circumscribed cones.* Draw at n a tangent ns to the meridian curve, and let the meridian curve and the triangle sin revolve about the axis si. The meridian curve generates the surface of revolution; the tangent sn, a cone tangent to it, having the circumference $n4p$ as its line of contact. Then the lines of shade of this cone will determine two points on the circumference of contact, which will be the points required.

Take the circle of contact np as the base of this cone. Through the vertex of the cone (s, o^h) draw a ray; t is its trace on B, the plane of the base of the cone; t^h4^h, t^h3^h are the traces on B of the planes of rays tangent to the cone; 3^h and 4^h are traces on B of the elements of shade; projecting these traces upon VB, we have 3 and 4 as the required points.

If the circle of the gorge G is assumed as the circle of contact, the auxiliary cone becomes a vertical cylinder. If two planes of rays be drawn tangent to this cylinder, the traces of the elements of shade on G will be λ^h and τ^h, which will be horizontal projections of points of the line of shade; λ^v and τ^v will be found in VG.

50. *Second method, by inscribed tangent spheres.* Assume the same circle of contact np, and at n^v draw a normal n^vo^v to the meridian curve; it meets the axis at o, which is taken as the centre of a tangent sphere. The circle of shade of this sphere is perpendicular to R; its trace on the plane of the principal meridian is x^vo^vr, drawn through o perpendicular to R^v; the circle of shade intersects the circle of contact B in a horizontal

right line perpendicular to R, and piercing the meridian plane at x; $3^h x^h 4^h$ is its horizontal projection, and the points 3 and 4, in which it intersects the circumference of contact of B, are the required points of the line of shade.

Remark 1. The point x^v may be determined thus: Draw $n^v y$ parallel to R^v, and yx^v parallel to $n^v s$; for the three perpendiculars dropped from the three angular points of the triangle $n^v y o^v$ upon the opposite sides must meet in the same point x^v.

Remark 2. When R^v and R^h are at 45° with GL, $iy = in^v$.

51. *Third method, by enveloping surfaces.* In Figs. 24 and 25 we have a sphere and a surface of revolution with its inscribed sphere (np being the assumed circumference of contact), both illuminated by the same system of parallel rays: the projections of the circumferences of shade on each sphere are ellipses similar both in form and position. Therefore drawing OP and PN respectively parallel to op and pn, joining OΓ and OΔ, we determine γ and δ by drawing $o\gamma$ and $o\delta$ parallel to OΓ and OΔ respectively.

A surface of revolution M may be considered as the envelope of the surface of a sphere S, the centre moving on the axis of M, and the radius varying according to a fixed law; the consecutive intersections of S are circumferences C, and the point in which the line of shade of S intersects the corresponding circumference C is a point of the line of shade of M.

52. PROBLEM 20. *Having given a portion of a surface of revolution convex toward the axis, it is required to find the line of shadow cast by the circumference of the upper base upon the surface.*

This problem is the continuation of Problem 19. Let the surface be given as in Fig. 31. *To find the highest point,* draw through b a line bp, making, with GL, the angle θ, which R makes with H, and meeting the contour line in r and the axis in p; draw through r a horizontal line, and through p a line parallel to R^v; the intersection 3^v is the vertical projection of the required point; its horizontal projection is 3^h upon kz^h, the horizontal trace of the vertical meridian plane of rays; for bp is the revolved position of the ray $3p$. To find other points of the curve, pass horizontal secant planes cutting the surface in circumferences W, X, Y, Z. The shadows of x on these planes are s, s_2, s_3, s_4; and the shadows of the circumference of the upper base (bd, $a^h n c^h$) upon these planes, are the circumferences W_2, X_2, Y_2, Z_2, whence we have, as their intersections, the points 1 2 4 5 6 7 8 9 as the points required.

53. *Pillet's method of casting shadows by means of a diagonal plane:* R^v and R^h at 45° with GL.[1]

For brevity, we will designate the plane P, Fig. 32, perpendicular to H, and making an angle of 45° with V as a diagonal plane.

The shadow on the diagonal plane P of a line ab, parallel to GL, is vertically projected in $a_2^v b_2^v$ at 45° with GL, found by drawing rays through a and b, and determining their traces $a^v{}_2$, a_2^h, b_2^v, b_2^h on P.

Let (Z, a_2) be the projections of an axis, and the distance of a point b in front of that axis be denoted by $\delta = a_2^h x^h$; then the point b_2^v may be found without using the horizontal plane, in the following manner: Draw a line through b^v parallel to GL, intersecting Z at x^v; lay off $x^v a_2^v = \delta$, and draw $a_2^v c^v$ at 45° parallel to R^h; draw $b^v b_2^v$ at 45°, and their point of meeting b_2^v is the required shadow: for the quadrilateral $b^v x^v a_2^v b_2^v$ equals $b^h x^h a_2^h b_2^h$. If b is behind the axis, then δ should be laid off above x^v, instead of below it, as in the figure.

54. *Shadow of a circle on the diagonal plane.*

Let the circle be parallel to H, Fig. 33, and let it be circumscribed by the square ad. Then we have the square $a_2^v d_2^v$ as its projection upon the diagonal plane and any side as $a_2^v b_2^v = ab \cos 45° = ab\sqrt{\tfrac{1}{2}}$: hence its surface is one-half of that of the original square, and the circle inscribed will be one-half of the original circle.

If the circle had the position indicated in Fig. 34, having the diagonal plane passing through its centre, the vertical projection of the shadow on the diagonal plane is a circle whose radius is the line $\nu_2 = ab \cos 45°$.

55. We shall apply this device in obtaining rapid solutions of a number of problems.

Application 1st. Let it be required, Fig. 45, to find the shadow of the circumference projected in bd on the surface of revolution. Imagine a diagonal plane drawn through the axis of the surface. Then the oblique projection of the circumference bd is obtained as follows: Draw bt at 45°, and zt

[1] This method is due to M. Pillet, Professor at the *École des Beaux Arts*, Paris.

perpendicular to it; with z as a centre, describe the arc tef: this is the required projection. In the same way the oblique projection of np is xys; they intersect at 1_2 and 2_2, which are the oblique projections of points of the shadow. Drawing lines $1_2 1$ and $2_2 2$ at 45° with GL, we have their intersections 1 and 2 with np as the points required.

56. *Application 2d. To find the line of shade of a cone.*

Let it be required to find the line of shade of the cone vertically projected in ade, Fig. 39, without using the horizontal projection. Project the cone on the diagonal plane P through its axis. Draw dd_2 at 45°, and od_2 perpendicular to it; with o as a centre and od_2 as a radius, describe a circle; this is the projection of the shadow of the base on P. The vertex a being in P is its own projection; therefore, drawing as_2 and ar_2 tangent to this circle, we have the vertical projection of the shadow of the cone on P. But the projections of lines of shadow must be the shadows of the lines of shade: hence, drawing the projecting lines $s_2 s$ and $r_2 r$ at 45°, we have ar and as as the required lines of shade of the cone.

57. *Application 3d. To find the shadow of a given point on the surface of a cone by the method of the diagonal plane.*

Let, Fig. b, Pl. V., abc be a cone, and n the projection of the given point situated at a given distance δ in front of the axis ao. The projection of the cone on the diagonal plane is ab_2c_2; that of the point n is n_2 [found thus: draw nt perpendicular to the axis; it meets it at t; set off $td = \delta$, and draw dn_2 at 45° parallel to R^h; draw nn_2 parallel to R^v]; join a with n_2, and produce it to e_2 on the circumference of shadow; the point e of the base, which casts the shadow e_2, is found by drawing ee_2 at 45°. Now join a with e, and ae is the element upon which the shadow of n is cast at n_1.

58. Problem 21. *To find the shadow cast upon a hollow cone by the complete circle of its base.*

Let the cone be given as in Fig. 38. For brevity, instead of saying the vertical projection of the shadow of a point a upon the diagonal plane, we shall say simply the projection of a upon P, or the oblique projection of a. Take the diagonal plane passing through its axis. Any plane of rays through o cuts two elements from the cone, one of which is the shadow of the other, and projects both on the diagonal plane in the same line: thus ok_2 and oz_2 are the vertical projections of an element and its shadow. Therefore, drawing a ray through a, we have x_2 as its oblique projection on the base of the cone; ox_2 is the oblique projection of oa, and ot_2 that of its shadow. Drawing $t_2 t$ at 45°, we have ot the shadow of oa; producing ax_2 to r, we have r as the shadow of the point a. Draw oy_2, oz_2, and oy; then, drawing rays through s and k_2, we have n and k as points of the line of shadow.

Remark. The lowest point n may *also* be found by drawing a line af, making an angle θ with az, and through its intersection s with oc a ray sn intersecting the horizontal through f at n. The point l projected on the axis is symmetrical with k; hence it is found at the intersection of a horizontal line through k with oc (Art. 47).

59. *An abridged solution of Problem 12* is shown in Fig. 40. The highest point b is found by drawing a line mo, making an angle θ with xf; it is the revolved position of a ray in the meridian plane of rays; making the counter revolution, the ray is vertically projected in ob; drawing rb parallel to xf, we have the highest point b. The diagonal plane cuts the column along ed, its line of shade, and the abacus in yk; dk is the oblique projection of the lower circle of the abacus, and d is the point common to this shadow and the line of shade; here the ray becomes tangent to the curve of shadow. The curve of shadow is tangent to sx, and falls between s and r; c is at the same height as a. Any intermediate points, if desired, may be found by the method explained in Art. 36.

60. Problem 22. *Shadow of an abacus and a quarter-round, or ovolo.*

As an example of the use of the diagonal plane, let it be required to determine the shadows in Fig. 36. This consists of an abacus pl^v, whose horizontal sections are squares, a quarter-round or semi-torus $10'e$, a fillet fg, and a cylinder vn.

1st, For *the line of shade of the ovolo*, the construction is the same as that of Fig. 23.

2d, *The shadow of this line* $\alpha\beta$ 45 on the diagonal plane through the axis zs^v, is the curve $\alpha\beta_2 y$, constructed by points as in Art. 53. α is in the diagonal plane; (the distance of β from the axis is $\beta 5$; laying off $\beta z = \beta 5$, and drawing a line

$z\beta_2$ parallel to R^h and a ray $\beta\beta_2$, we have β_2 at their point of intersection; or drawing $5\beta_3$, we have also $\beta\beta_3 = \beta\beta_2$). Again, the tangent at β_2 makes an angle of 45° with mn: hence we have the directions of three tangents to the curve; viz., at α, 90°, at β_2, 45°, and at y, 0° with mn.

3d, *The shadow of the edge of the abacus pq on the ovolo.* This line casts a shadow qw on the diagonal plane. The shadow of pq on the ovolo is the section by a plane parallel to mn and inclined to V at an angle of 45°. If this plane rotates 45° about the axis of the capital, it becomes perpendicular to V, and will be entirely projected along its trace pw, which will be the shadow of the edge vertically projected at p. (To see how this plane revolves, imagine (Fig. 4, Pl. I.) R to revolve 45° about a vertical line passing through its centre: o would pass to o^v, c to t^h, and the plane oo_0ct would become o^vct^ho, having R^v as its vertical trace. The shadow of oo^v is o^vc.)

We have the highest point, 10, of this shadow by projecting $10'$ on zs^v.

The points 2 and 3 are obtained by taking the intersections 2_2 and 3_2 of the auxiliary shadows wq and $\alpha\beta_2y$, and projecting back by reversed rays upon the line of shade of the ovolo. These two extreme rays, since they are tangent to the ovolo and lie in the intersecting plane of the curve 2, 10, 3, must also be tangent to the curve at the points 2 and 3 respectively.

Remark 1. By symmetry, the point 1 is another point of the line of shadow.

Remark 2. The edge of the abacus projected at p, having pw for its shadow, meets the line of shade at 16; hence 16 and 3 are at the same height, for both are symmetrically situated with respect to $4y$.

4th. *The shadow of the abacus on the column.* With w as a centre and a radius equal to zm, describe an indefinite arc 67; it is the indefinite shadow of the edge pq on the column. By Art. 59 we determine the shadow of $\beta345$ on the column. $12y$ makes an angle θ, and determines 13; 14 and 15 are at the same height; 8 is determined by the intersection of the line of shade $r8$ with the curve $2_2\beta_23_2y$; the intersection of the two shadows at 7 gives the final shadow on the column, ν, 6 7, 15, 8.

Remark 3. We have found the shadow by the method of the diagonal plane; but as a verification, and to show the accuracy of the method, we have drawn the plan, and passed a vertical secant plane of rays through any point, as b, taken at random on the edge of the abacus: this plane has $b^h\omega^h$ as its horizontal trace. It cuts from the abacus the line (b^vc, b^h); from the ovolo a curve di (e, o^h); from the fillet (eo, o^h); and from the column $(x\omega^v, \omega^h)$; (the point i is found by intersecting the surface by a horizontal plane; X^v and X^h are the horizontal circumferences it cuts from the surface, and i is the point in which X meets the vertical plane of rays). It is seen that c gives the point c_1, and τ the point 9.

61. Problem 23. *To find the shades and shadows on the base of a Tuscan column.*

The base of such a column, represented in elevation in Fig. 37, consists of three parts: viz., 1°, a rectangular prism gk, called a plinth, whose horizontal sections are squares; 2°, a torus mf; 3°, a cylindrical fillet rn. The shaft is concave outward as far up as b: it then becomes nearly cylindrical; and, in casting the shadows, it is considered as a cylinder above b.

The line of shade px of the shaft casts a shadow upon the surface formed by the revolution of bsd called the conge, and also on the torus. To find any point of this shadow, we make use of the diagonal plane. This plane passes through zo and px. Any horizontal plane ts cuts a circumference from the conge, a part of which is projected on the diagonal plane in the arc Vy, intersecting px in y; projecting y on ts by the ray $y\epsilon$, we have ϵ as a point of the shadow of px on the conge; the shadow on the torus can be determined in the same manner. These shadows consist, 1°, of the shadow of the line of shade 1 2 of the fillet; 2°, of that of the arc 1 4; and, 3°, of a portion of py; and are represented by the curve 1_14_15.

SHADOWS ON SLOPING PLANES.

62. Problem 24. *To find the shadow of a given point on a given plane parallel to the ground line.*

Let, Fig. 41, a be the given point, and let the given plane M pass through GL, and make a given angle α with the horizontal plane H. Let R^v and R^h make 45° with GL. Pass a profile plane through a, and revolve it about its vertical trace $a^v a_0$ until it coincides with V; a is found at a'; $a_0 m$ is the revolved position of the trace of M on the profile plane P. Since the projection of R on P makes 45° with $a^h a_0$, it has $a'x'$ (making 45° with GL) for its revolved position; making the counter revolution, x' returns to x, which is the projection on P of the shadow of a on M: this shadow therefore lies in a perpendicular xy to $a^h a_0$, and also in $a^v y$, the vertical projection of the ray through a: hence it lies at their intersection y. Through y draw bn parallel to $a_0 m$; then, since $a_0 b = a_0 a^h = \delta$, we have the following rule for determining immediately y, knowing a^v, α, and δ.

RULE. From a_0 set off $a_0 b = \delta$; draw bn, making the given angle α with GL, and through a^v a right line at 45° with GL: its intersection y with bn is the shadow required.

63. PROBLEM 25. *To find the shadow of a given horizontal circle on a plane parallel to the ground line, and making a given angle with the horizontal coördinate plane.* R^v and R^h at 45° with GL.

Let (Fig. 44) ad be a square circumscribing the given circle, and let α be the angle of the plane with H. The shadow of the centre o is found at o_1 by taking $o_0 c' = o_0 o^h$; $c'b_1$ making an angle α with $a'c'$, and $o^v s$ at 45° with GL; $r_1 t_1 = ab$; $c_1 b_1 = b^v y = ab \cos \alpha$; whence we have the parallelogram $a_1 b_1 d_1 c_1$, and $su = ab \sin \alpha$; the inscribed ellipse is the shadow of the given circle.

64. PROBLEM 26. *To find the shadows cast by the chimneys and dormer window upon a sloping roof.*

Let the roof be given as in Fig. 42; $qrp = \alpha$, the angle of the roof with H.

1st, *The shadow of the chimney.* Fig. 43 shows the plan of the chimney. Make $l_0 m_3 =$ twice the distance of m in front of l; lay off $o_0 e$ and $o_0 u = o_0 o'$, and $o_0 o''$, Fig. 43, and through y draw a ray yy_1, and through e and u lines parallel to rq, meeting yy_1 in y_1 and y_2; y_1 and y_2 are the shadows of the vertices of the conical chimney-pots, and o_1 and o_2 the centres of the shadows of their upper bases: the shadows of these bases are ellipses found by Problem 25. Through l_0 and m_3 draw lines parallel to rq; $l_0 l_1$ is the shadow of $l_0 l$; m_1 is the shadow of m; $m_1 n_1$ is the shadow of mn, and $n_1 o_2$ is the shadow of the edge vertically projected at n. Lay off $vx = ms$, Fig. 43, and draw xz_3 parallel to qr; drawing a ray through t, we have t_1 as the shadow of t, and $t_1 o_2$ parallel to ab, as the shadow of the edge us, Fig. 43.

2d, *To find the shadow of the dormer window,* draw $g'l'$; then 1 is situated at a distance $4_0 1_0$ in front of $g'l'$, and also the same distance to the right of 4_0; hence, setting off $4J =$ twice $4_0 1_0$, and drawing $4_0 4_1$ and $J1_1$, and also rays through 1, 2, 3, 4, we have $4_0 4_1$, the shadow of $4_0 4$, and $1\,1_1$ the shadow of the edge vertically projected at 1; in the same way the points 2_1 and 3_1 are determined.

65. PROBLEM 27. *To determine the shadow of a cornice.*

Fig. 30, Pl. IV., is the elevation of a cornice. To find the shadows cast, draw $ca' = ga'$; $em' = ym'$; $tv' = zv'$; then lines drawn through a', m', v', parallel to ac, are the shadows of ac, me, and uv respectively.

THE HELICOID.

66. *Properties of the warped helicoid.*

We have seen (Arts. 234, 235, *Des. Geom.*) how to assume a point on the surface, and pass a plane tangent to the surface at that point. Fig. 49 is reproduced from Fig. 137 (*Des. Geom.*), with a few omissions. oep is the directing helix, (Ak^h, fk^v) an element making a constant angle α with the axis, a the point of contact of a plane tangent to the helicoid through this element, and $d^h k^h$ the horizontal trace of this tangent plane. Denote the pitch of the helix by p. Draw through a^h the right line ty perpendicular to $k^h d^h$, and erect at A a perpendicular At to Ak^h. The two triangles $d^h a^h k^h$ and $a^h At$ are similar, since their sides are perpendicular: hence,

$$\frac{a^h d^h}{a^h k^h} = \frac{Aa^h}{At}: \qquad (1)$$

d^h is the horizontal trace of the tangent to the helix passing through a; hence

$$\frac{a^v q}{a^h d^h} = \frac{p}{2\pi \times Aa^h}; \qquad (2)$$

both expressions being the tangent of the angle made by the helix with H. Multiplying (1) by (2), we have $\frac{a^v q}{a^h k^h} = \frac{p}{2\pi \times At} = \cot\alpha$, whence $At = \frac{p}{2\pi}\tan\alpha = c$. If we put $\frac{p}{2\pi} = h$ the reduced pitch, we have $c = h \tan\alpha$.[1]

Hence, to draw a plane tangent to a warped helicoid at any point a of a given element fk, it is sufficient to draw $At = c$ perpendicular to Ak^h, to join t and a^h; then the perpendicular $k^h y$ dropped from k^h upon ta^h produced, is the horizontal trace of the required tangent plane.

Again, if the trace $k^h d^h$ of the tangent plane containing the element fk be given, to find the point of contact of this plane, it is sufficient to erect the perpendicular $At = c$ to Ak^h, and drop ty upon $k^h d^h$; ty intersects Ak^h at a^h, the horizontal projection of the required point of contact.

If the point of contact a moves along the element fk, $d^h k^h$ and At rotate around k^h and A respectively, and when a reaches k, $d^h k^h$ becomes tangent to the spiral $o^h k^h$, and the normal at k^h will pass through the revolved position of t.

[1] This relation is immediately obtained from the equation of the base of the helicoid; viz., $r - R = c\phi$ given in the note to Art. 234, *Des. Geom.*; for differentiating $\frac{dr}{d\phi}$ = subnormal $= At = c$.

67. Problem 28. *To find the line of shade on the surface of a given helicoid.*

Let aye, Fig. 46, be the directing helix, D a given element making a given angle α with the axis (sf, o). The direction of the light indicated by the arrows R^v, R^h, is assumed at an angle θ with the horizontal, and parallel to the vertical coördinate plane. The shadow of D is D_1, found by drawing a ray of light through any point, as (o, f); this ray pierces H at f_1^h, which, joined with a^h, gives D_1.

From s lay off on the axis, $sx = h = \frac{p}{2\pi}$, and make sxt^v equal to the given angle α, which every element makes with the axis. Draw xr'' parallel to R^v, so that the angle $xr''s = \theta$; then $h \tan\alpha = c$; denote $sr'' = h \cot\theta$ by m.

Now, the plane of rays passing through D and D_1 is tangent to the helicoid at some point of D. To find this point, erect at o a perpendicular od to D^h, and make $od = c$; then from d drop a perpendicular on D_1, and the point z^h, in which it meets D^h produced, is the horizontal projection of the point of contact z of the plane of rays with the helicoid, i.e., a point of its line of shade. Draw through z^h and d lines parallel to GL; produce D_1 to z_1 and pd to r. Denote the angle $k^h oa^h$ by ϕ, and the distance oz^h by ρ. The triangles rod and $a^h of_1^h$ have their sides respectively perpendicular: they are therefore similar, and we have $\frac{ro}{of_1^h} = \frac{od}{oa^h}$; but $of_1^h = sf_1^v = sf\cot\theta$; $oa^h = sf\tan\alpha$; $od = c$; hence $ro = sf\cot\theta\,\frac{c}{sf\tan\alpha} = \frac{c\cot\theta}{\tan\alpha} = h\cot\theta = m$.

The point r is called the *centre of radiation*, and is a fixed point for the same values of h and θ. With o as a centre and $od = c$ as a radius, describe the circumference $t^h d$: then, with r and the distance c, we can at once find the horizontal projection of the point of contact z^h for any element whatever, as D^h, by erecting a perpendicular od joining d, its intersection with the circumference t^hd, with r; the point z^h, in which dr crosses D^h, is the point required. In this way the curve $1\,0\,3\,2r\,0z^h4$ is obtained.[1]

68. *To find the point of the line of shade situated upon any given helix traced upon the surface.*

Given c, m, the position of r, and the base aek, Fig. 50, of a helix, it is required to find that point of the line of shade which touches this helix. From the centre o, with c as a radius, describe a circle $d'd''d'''$. Let r be the centre of radiation, ro being taken equal to m; draw roa; draw ke perpendicular to it; join ad', and with o as a centre and m as a radius, describe an arc rr', meeting ad' produced in r'. Now, if the whole system revolve about a vertical axis at o until r' coincides with the fixed point r, $ad'r'$ will take the direction $1\,dr\,2$, and the point 1 is the horizontal projection of the shade on the element $o1$, making with ok an angle ϕ: 2, situated on *another* element $o\,2$, is a second point. In the same way, by drawing ad'' and revolving it in the contrary direction, we obtain the points 3 and 4, all, however, on different elements. Comparing Figs. 46 and 50, we see that the points of shade are situated on lines passing through r symmetrically disposed with respect to ao, so that when one point has been found the others can be immediately determined.

Join o with r', and produce it to l; then evidently $lok = ko\,1 = \phi$,[2] which, with the assigned value of ρ, determines the point 1.

[1] The implicit polar equation of this curve is at once obtained: thus, draw dl parallel to GL; then by the similar triangles rnz^h and rld, we have $\frac{nz^h}{ld} = \frac{rn}{rl}$ or $\frac{\rho \cos \phi}{c \sin \phi} = \frac{m - \rho \sin \phi}{m + c \cos \phi}$. In Cartesian coördinates,

$$x = \rho \cos \phi \qquad \rho = \sqrt{x^2 + y^2} \qquad \sin \phi = \frac{y}{\sqrt{x^2 + y^2}}$$

$$y = \rho \sin \phi \qquad\qquad \cos \phi = \frac{x}{\sqrt{x^2 + y^2}}$$

Substituting these values in the equation, we have $\frac{x\sqrt{x^2 + y^2}}{cy} = \frac{(m - y)\sqrt{x^2 + y^2}}{m\sqrt{x^2 + y^2} + cx}$; whence $mx\sqrt{x^2 + y^2} = c\,[my - (x^2 + y^2)]$, or $m^2x^2\,(x^2 + y^2) = c^2\,[(my - (x^2 + y^2)]^2$.

[2] Put $oa = \rho$; $d\,1\,o = \gamma$; $\tan \gamma = \frac{c}{\rho}$; (1)

$$od = c\,;\quad ko\,1 = \phi\,;\quad \frac{\rho \sin \gamma}{m} = \sin \varepsilon\,; \qquad (2)$$

$$or = m\,;\quad or\,1 = \varepsilon\,;\qquad \phi = 90^\circ - \gamma - \varepsilon. \qquad (3)$$

The equations (1), (2), (3) must be fulfilled for every value of ρ, i.e., for every helix traced on the surface.

69. Fig. 47 shows this construction with the values of c, m, and ρ, taken from Fig. 46. The triangle $txr = t^vxr''$; $sa = \rho = ok^h$; join a and t, and with s as a centre and a radius $sr = m$, describe the arc ro meeting at produced in o; join s and o, and with a radius $= sl = ok^h$ describe the arc lu; then $lu = k^h1^h$, and determines the point 1. It is evident that lu, in Fig. 47, is constructed in exactly the same manner as lk in Fig. 50.[1]

70. Problem 29. *To construct the shades and shadows on the different parts of a screw, and the shadow cast on the horizontal coördinate plane.*

If the vertices of an isosceles triangle move around an axis situated in the plane of the triangle, and parallel to its unequal side so as to describe three helices of the same pitch,[2] the equal sides of the triangle will describe the surface of the triangular threaded screw.

This surface, represented in Fig. 51, is generated by the isosceles triangle ($l^vn^vl_2^v$, l^hn^h) moving around the axis (Z^v, o) so that the vertices describe the three helices lp, ns, and l_2p_2. The direction of the light is denoted by R.

If we attempt to apply directly the method explained in Art. 69, we shall find the values of c, h, and m to be so small as to preclude their use in the construction: to obviate this difficulty, we shall use these constants multiplied by π. This gives $\frac{p}{2} = \frac{\text{pitch}}{2} = \pi h = \frac{n^vn_2^v}{2}$. Therefore, drawing an indefinite horizontal line ur, Fig. 52, and erecting a perpendicular $sx = \frac{n_2^vn_2^v}{2}$, and drawing st parallel to l^vn^v, and xr parallel to R$'$, we have $ts = \pi c$ and $rs = \pi m$; take $sa = \pi \times ol^h$, and draw ato; with s as a centre and a radius sr, draw the arc ro; then with s as a centre and a radius $sl = ol^h$, describe the arc lu. Draw, in Fig. 51, ou parallel to R^h, and from u lay off the arc $u4^h = lu$, Fig. 52, and we have the point 4^h of the line of shade, whence we obtain the vertical projections 1^v, 4^v on the vertical projection of the helix. The points 2, 3, and 5 are obtained in the same manner. Drawing oX perpendicular to ou, the points 7^h and 8^h will be symmetric with 4^h and 5^h.

[1] Dropping the perpendicular sz upon at, we have $\tan \gamma = \frac{c}{\rho}$ (1), $\frac{\rho \sin \gamma}{m} = \sin \varepsilon$ (2), and $\phi = 90^\circ - \gamma - \varepsilon$ (3).

[2] The two vertices adjacent to the unequal side describe different spires of the same helix.

SHADOWS.

The shadow of the screw on the coördinate plane H is limited by that of the outer helix, which is determined by points; e.g., the points 4_1, 7_1, and 9_1 are the shadows of 4, 7, and 9.

Shadow cast on the surface by the curve of shade. 1st, To find the shadow cast upon the surface by the curve 4, 5, we use the method of oblique projections (Art. 44); the shadows upon H of the curves 4 5 and 7 8 are $4_1\, 5_1$ and $7_1\, 8_1$; these, as shown in Fig. 54, intersect in a point y_1; therefore, drawing a ray through y_1, we determine the point y_0, which casts its shadow y upon 7 8. The curves $4\,y_0$ and $y\,7$ have real shadows $y_1 4_1$ and $y_1 7_1$.

The curve 7 8 is not illuminated. The curve $5\,y_0$ casts the shadow $5^v\, d^v\, y$ on the upper fillet. *To find any intermediate point as d^v*: Draw an element ae; its shadow is $e^h b_1 a_1$, intersecting $4_1\, 5_1$ at d_1, which, projected back by a reversed ray on to ae, gives the required point d^v, the shadow on the upper fillet of a point of the curve 4 5.

71. *Shadow of a helix.* Fig. 48 shows the oblique projection or shadow of a helix. Imagine the helix to be illuminated by rays whose direction is indicated by the arrows; the shadow of any point n is n_1[1]; draw $a_2^v s = \dfrac{\text{pitch}}{4}$; $a_2^v k_0$ parallel to R^v; sq parallel to $k_0 a^v$; $bd = \frac{1}{2}$ arc $bn^h a^h$; join $o^h d$ and project q upon it at e; draw ef, then the curve $a^h n_1 k$ is a curtate cycloid described by the point a^h, carried around by the circle iqf rolling along the line ii'.[1]

72. In Fig. 53, the form of several spires of the shadow of the outer helix of the triangular threaded screw is shown. By Art. 71, $os = h$ cot θ; but, Fig. 52, $sr = \pi h \cot \theta$; and $sa = \pi\rho$; if we take $ai = \rho = ou$, Fig. 51, we have $\dfrac{\pi h \cot\theta}{\pi\rho} = \dfrac{ii'}{\rho}$; or $ii' = h \cot \theta$.

73. To show the form (not the position) of the shadow of the outer helix pbl, Fig. 51, we choose at random a centre o, Fig. 53, and with a radius ii', Fig. 52, describe the circle smz; draw ol and os parallel and perpendicular respectively to R^h; draw sg parallel to ol, and $o\nu$ parallel and equal to $y\nu$, Fig. 51[2]; erect at ν an indefinite perpendicular; with o as a centre and a radius ou, describe an arc meeting νn in n; n is the initial point of the displaced shadow (its real position being obtained by moving the figure parallel to itself until o coincides with o, Fig. 51); the curve is described by the extremity of the radius vector on when the circumference zms rolls along the line sg.

[1] The equation of the curve may be deduced thus: Assume a^h as the origin, $a^h k$ and $a^h b$ as the axes of X and Y respectively; put $r = o^h b$; $h = \dfrac{a^v a_2^v}{2\pi}$ = the reduced pitch; $\theta = a_2^v k_0 a^v$, the angle of the light; $\phi = a^h o^h n^h$. Then, for the shadow of any point n, we have n_1, whose coördinates are $x = n_1 t = [n_0 o_0 = n\, o^v = n^h t] + o_0 a^v$, or $x = \phi h \cot\theta - r \sin\phi$; $y = r + o^h t = r - r\cos\phi$. This is the equation of a curtate cycloid, in which r is the describing radius, and $h \cot \theta$ that of the rolling circle.

[1] For by construction $a_2^v s = \dfrac{p}{4} = \dfrac{h\pi}{2}$; $ef = \dfrac{h\pi}{2}\cot\theta$; $db = \dfrac{r\pi}{2}$: since $\dfrac{db}{ef} = \dfrac{bo^h}{fo^h}$, we have $fo^h = h \cot \theta$. The angle $bo^h d = \tan \dfrac{\pi}{2} = 57° \, 31' \, 06''$.

[2] ν is the vertical projection of the horizontal trace of the outer helix pbl, found by producing $p^v b^v l^v$ until it meets GL.

ABRIDGED CONSTRUCTIONS OF THE PROBLEMS OF SHADES AND SHADOWS.

R^v AND R^h AT 45° WITH THE GROUND LINE.

[The theoretical explanations of the constructions, having been already fully given, are omitted.]

Fig. 0. B represents an opaque body separated by the curve C into the illuminated portion and the shade; C is the line of shade, M the shadow on S, and C_1 the line of shadow.

Fig. 1. To find the shadow of a on V. Denote Ga^h by δ. Lay off $a^v o = \delta$, draw through o a line parallel to GL, and set off on it a distance $oa_1 = \delta$: a_1 is the shadow required.

Fig. 2. To find the shadow of c on H. Denote $c^v n$ by δ. Lay off $c^h x = \delta$, draw through x a line parallel to GL equal to δ, and its extremity c_1 is the shadow required.

Fig. 3. To find the shadow of A on V. a_1 and b_1 are the shadows of a and b, and the line $a_1\, b_1$, or A_1, is the shadow required.

Fig. 4 shows the conventional direction of the ray of light R to be the diagonal oc of a cube so placed as to have two of its faces parallel to the vertical, and two to the horizontal plane of projection. Consequently R^v and R^h make angles of 45° with the co_o; again, R′, the projection of R on the profile plane oo_o, makes an angle of 45° with $o_o o^h$.

Denoting oco^h by θ, we have approximately $\tan\theta = 0.3$; $\operatorname{cosec}\theta = 1.73$; $\sin\theta = 0.577$; and $\cos 2\theta = \frac{1}{3}$ exactly.

Fig. 5 shows that the angle θ is easily constructed by assuming a point (o^h, o^v) on the ray (R^h, R^v), and revolving this ray around cx until it coincides with the vertical plane V, whence we have $oc\mathrm{G} = \theta$.

Fig. 6. Let $abcd$ be a square parallel to V: it is required to find its shadow on H. Denote the given distance tc^v by δ, and the length of the side $a^h b^h$ by l; make $a^h n = \delta$, $no = os = l$; erect at n, o, and s, three perpendiculars to GL, and lay off on the first the distance δ, on the second the distances δ and $\delta + l$, and on the third the distance $\delta + l$: the extremities of these perpendiculars will be the angular points sought; viz., c_1, d_1, a_1, b_1, whence the shadow is known.

Fig. 7. To find the shadow of A on V. Let $tA^h = \delta$. Lay off $tx = \delta$, and erect a perpendicular A_1 to GL at x; A_1 is the shadow of the unlimited line A.

Fig. 8. To find the shadow of B on V. The shadow of any point of B as (o^h, B^v) is o_1, which joined with B^v gives the required shadow B_1.

Fig. 9. To find the shade and shadow of the prism on V. The visible shade is $c^v d^v ln$. The points o, p, q, and r are found by laying off on GL the following distances: $mo = mb^h$; $np = nc^h$; $lq = ld^h$; and $nr = ne^h$; erecting perpendiculars to GL at o, p, q, r, and drawing the vertical projections of rays through b^v, c^v, d^v, their intersections with the respective perpendiculars give the points $b_1 c_1 d_1 e_1$, which, with $e_1 r$, determine the shadow on the vertical plane.

Fig. 10. To find the shade and shadow of the cone. Draw a ray through the vertex o; o_1 is its horizontal trace. Drawing $o_1 n^h$ and $o_1 c^h$ tangent to the base of the cone, we have the horizontal traces of the two tangent planes of rays; the lines of contact are oc and on; hence noc is the line of shade. The portion of the shade visible on the horizontal projection is $o^h l c^h$; that visible on the vertical projection is $o^v l_0 n^v$. The shadow on H is $c^h o_1 n^h$.

Fig. 11. To find the shade and shadow of a sphere. Denote the diameter of the sphere by d: then $qs = on = d \sin\theta = 0.577d$; $o_1 n_1 = d \operatorname{cosec}\theta = 1.732d$. Fig. a shows that $cs = oc \sin\theta$ $= c^v s$, Fig. 11; $oc = \dfrac{d}{2} = oa^v = oa^h$. $\phi = \theta$.

Fig. 12. To find the shadow of a niche upon its interior surface. The line of shadow is partly composed of the arc of an ellipse extending from f, the point of contact of the tangent ray, to x, having for its semi-axes zf, the radius, and $zo = \frac{1}{3} zf$; any point of xd_1^v as s_1^v may be found by passing a ray $(s^v s_1^v, s^h s_1^h)$ through s, and finding its trace (s_1^v, s^h) on the surface of the cylinder; $d_1^v n_1$ is the shadow of dn, a portion of (da^v, a^h), upon the interior surface of the cylinder; n is found by drawing $n_1 n$ parallel to R^v.

Fig. 13 shows the shadow of the edge of a hemispherical shell upon its interior surface, to be a semi-ellipse having ec^h and $c^h s^h = \dfrac{ec^h}{3}$ for its axes when $\theta = 35° 14' 52''$.

Fig. 14. To find the shadow of the circle *kfo*. The shadow of c is c_1; the shadows of the diameters *ke* and (gf, f_1) are k_1e_1 and f_1g_1, two conjugate diameters of the ellipse of shadow. Taking $k^vn = gf$, joining n with c and producing it to o, *no* and *nx* are equal to the axes of the ellipse of shadow s_2s_1 and z_2z_1; also $s_2c_1e_1 = \gamma = \frac{1}{2}nck^v$.

Fig. 15. To find the shadow of the circle in the profile plane given by its centre c, and its diameter $fg = ek$. The shadow of c is c_1, that of the horizontal diameter (ek, c^v) is e_1k_1, that of the vertical diameter (c^h, fg) is c^hg_1; s_1s_2 and z_1z_2 are found from Fig. 14.

Fig. 16. To find the axes a and b of an ellipse when a', b' and ω are given. From the extremity n, of one diameter, drop a perpendicular *nf* on the other produced; from n set off distances $nl = nr = ce$; join c with l and r, and draw through n a parallel *nd* to *cr*, intersecting *cl* in o; make $ox = od = ol$, and join c with d and x; make $ca = nd$, and $cb = nx$, then *cb* and *ca* will be the new axes both in magnitude and position.

Fig. 17. To find the shade of the cylinder, and the shadow of its upper base on the interior surface. Denote the radius of the base of the cylinder by ρ; then $bp = \rho$ versin $45° = \frac{3}{10}\rho$[1]; $c^ve_1 = \rho$; $c^vy_1 = \rho\sqrt{2}$; the angle $y_1c^ve_1 = 45°$; making $e^vn = 2\rho$, and with c^v as a centre describing the semi-circumference, xe^vo, *no*, and *nx* are the lengths of the axes of the ellipse of shadow, and the angle $nc^ve^v = 2\gamma$.

Simplified construction. Lay off $c^vt^v = c^vr^v = \frac{7}{10}\rho$, and draw the elements r^vp and c^vs; set off $c^ve_1 = c^ve^v$, and $c^vg = 2c^vt^v$; draw e_1y_1 and gl_1 parallel to GL; their intersections with f^vb and r^vp determine y_1 and l_1. We then have the five points t^v, e, l_1, y_1, and r^v, and their corresponding tangents t^vg, e^ve_1, gl_1, f^vy_1, and a right line through r^v parallel to t^vg.

Fig. 19. To find the shadow of a rectangular abacus on a cylindrical column, and on V. Pass a plane of rays through the axis of the column; c^hs^h is its horizontal trace; this plane intersects the edge *ab* in s, and s^vo is the vertical projection of the ray through s. Denote the length of the radius of the column by ρ.

With o as a centre and ρ as a radius, describe an arc s_1^vny; this arc will be the vertical projection of the shadow of *ab* on the column; take $zr^v = \frac{3}{10}\rho$, and $zx_1 = c^vz$; erecting a perpendicular at x_1, and drawing *yu* at 45°, we have u_1. Drawing rays through b and e, we complete the outline by drawing u_1b_1 parallel, and b_1e_1 perpendicular, respectively, to GL.

To find the shadow of a given point a on a cylinder. Draw through a^v a line parallel to GL, meeting the axis of the cylinder in b'. Set off $b'o = a^hi$, the distance of a in front of the axis: o is the centre of s_1^vny, and the radius is that of the cylinder. Drawing through a^v a line parallel to R^v, we have the shadow a_1 of a upon the cylinder, without drawing any line on H. If the cylinder were hollow, and the shadow were required upon the interior surface, $b'o$ would be laid off above a^vb', instead of below it.

Fig. 20. To find the shades and shadows of the abacus and column in Fig. 20.

Let the column be intersected by vertical planes of rays cutting the lower base of the abacus in points a, b, d, etc., and the surface of the column in elements horizontally projected in α^h, β^h, δ^h, etc.; drawing rays through a, b, d, etc., of the abacus, we have their corresponding shadows α, β, δ, etc.

The lines of shade and shadow of the column are determined by its tangent planes of rays: these lines are ($\epsilon^v\epsilon'$, ϵ^h) and (ϵ_1o^v, o^h): ($w^vw_2^v$, w^h) is the line of shade of the abacus. The line of shadow of the abacus on V, cast by the arc ($y^vw_2^v$, y^hw^h), is a portion of an ellipse beginning at y and extending to α: here it is intercepted by the column and its shadow as far as ϵ_1, where it begins again, and extends to w_2; w_1w_2 is the shadow of ($w^vw_2^v$, w^h); w_1n^v is the shadow of the upper edge of the abacus *wn*.

Fig. 21. To find the brilliant point d on a surface of revolution. *ao* is the incident, *oz* the reflected ray, and *ob* the line bisecting the angle *aoz*. *on* and *op* are respectively perpendicular to a^vo and ob^h. *ce*, perpendicular to *op*, gives r^h, which revolves to r'; nr' is parallel to the tangent to U^v, passing through the revolved position d'' of the brilliant point d.

Fig. 26. To find the brilliant point on a spherical surface.

Fig. 26 represents a quadrant of a sphere, the centre c being in GL; R a ray of light to the centre c; R′ is the position of the ray revolved to coincide with H; *cn* is the bisecting normal in its revolved position; and d the real position of the required brilliant point.

Fig. 22. To find the line of shade of a torus.

Points on the contour lines X^v *and* Y^h are found by drawing tangents to these lines parallel respectively to R^v and R^h; thus determining the points α^v, ϵ^v, β^h, η^h, in which the projections of the line of shade are tangent to the contour lines of the torus: α^h, ϵ^h, β^v, and η^v are found on X^h and Y^v respectively; drawing tangents making an angle θ with GL, we have the revolved position x and z of the highest and lowest points; making the counter revolution, x and z take the positions ν and δ.

The arcs $\alpha\lambda$ and $\epsilon\gamma$ determine the points λ and γ on the profile meridian.

Fig. 23. *Abridged method for finding the visible line of shade on the vertical plane of projection.*

[1] If the diameter is $0^m.1$, then $\frac{3}{10}$ of $0^m.05$ is accurate to within $0^m.0005$.

Tangents to the vertical contour line parallel to R^v determine the points α and ϵ; $\epsilon\gamma$, drawn parallel to GL, determines γ at its intersection with the axis *on*. Draw to the contour a tangent *zn*, making an angle θ with *om*, and produce it to the axis at *n*. *z* is the revolved position of the lowest point; making the counter revolution, *n* remains in the axis, and *nz* takes the direction $n\delta$, parallel to R^v; drawing $z\delta$ parallel to *om*, we have δ as the intersection of $n\delta$ and $z\delta$. We have then five points and three tangents to $\alpha\beta\gamma\delta\epsilon$, which are abundantly sufficient for its construction in practice.

Fig. 27. To find the line of shade of any surface of revolution. Assume *np* as the circle of contact of the surface with that of an inscribed sphere, and at n^v draw a normal n^vo^v; lay off $iy = in^v$, and draw yx^v perpendicular to n^vo^v. x^h is found on e^hf^h. Through x^h draw a line perpendicular to R^h, intersecting the circumference $n^h6^hp^h$ in the points 3^h and 4^h, which are the horizontal projections of 3 and 4, the points required.

In Figs. 24 and 25 we have a sphere and a surface of revolution with its inscribed sphere (*np* being the assumed circumference of contact), both illuminated by the same system of parallel rays: the projections of the circumferences of shade on each sphere are ellipses similar both in form and position. Therefore drawing OP and PN respectively parallel to *op* and *pn*, joining OΓ and OΔ, we determine γ and δ by drawing $o\gamma$ and $o\delta$ parallel to OΓ and OΔ respectively.

Fig. 31. To find the shadow of the circle (bd, a^hnc^h) on the surface of revolution. *To find the highest point*, draw through *b* a line *bp*, making, with GL, the angle θ, which R makes with H, and meeting the contour line in *r* and the axis in *p*; draw through *r* a horizontal line, and through *p* a line parallel to R^v; the intersection 3^v is the vertical projection of the required point; its horizontal projection is 3^h upon kz^h, the horizontal trace of the vertical meridian plane of rays. To find other points of the curve, pass horizontal secant planes cutting the surface in circumferences W, X, Y, Z. The shadows of *x* on these planes are s, s_2, s_3, s_4; and the shadows of the circumference of the upper base (bd, a^hnc^h) upon these planes, are the circumferences W_2, X_2, Y_2, Z_2, whence we have, as their intersections, the points 1 2 4 5 6 7 8 9 as the points required.

Shadows of Points on Curved Surfaces.

Fig. 10. To find the shadow of the point *a* on the cone. Draw rays through *a* and *o*; a_1 and o_1 are their horizontal traces, and o_1s the trace of their plane: this plane cuts from the cone the element *os*; which intersects the ray through *a* at the point α, the required shadow.

Fig. 28. To find the shadow of a point, *a*, on the surface of a sphere.

Pass through *a* a plane of rays perpendicular to V, intersecting the sphere in a circle having *xz* for its diameter; rotating this plane about its vertical trace a^vz until it coincides with V, $xa_1'z$ is the revolved position of the circle, and $a'z$ that of the ray. $a^va' = a_oa^h$; a_1' is the revolved position of the shadow of *a* upon the sphere; making the counter revolution, a_1' returns to a_1, which is the shadow of *a*.

Fig. 29. The line of shade of an inscribed and circumscribed surface. For example, let, Fig. 29, a cone and cylinder circumscribe a sphere having the circumferences *bc* and *ef* as the respective lines of contact. Then the plane of rays tangent to the cone along *an* must be tangent to the sphere at *n*, and, therefore, *n* is a point of the line of shade of the sphere: for a similar reason, *s* is also a point of the same line of shade.

Remark. Fig. 29 illustrates the fact that when two surfaces are tangent, the lines of shade are not therefore tangent, as *an* and *st* make acute angles with *rsnk*.

Fig. C. *The method of oblique projections.* Let, Fig. C, Pl. V., C and D be two curves in space. It is required to find the point on C which casts its shadow on D. For this purpose we find, on an auxiliary plane P, the shadows C_1 and D_1 of the two given curves, which are oblique projections of C and D. The point of intersection x_1 of C_1 and D_1 is the trace on P of the ray which meets C and D; its point of meeting *x* on C and x_2 on D are the points required.

Fig. 30, Pl. IV., is the elevation of a cornice. To find the shadows cast, draw $ca' = ga'$; $em' = ym'$; $tv' = zv'$; then lines drawn through a', m', v', parallel to *ac*, are the shadows of *ac*, *me*, and *uv* respectively.

Pillet's method of casting shadows by means of a diagonal plane.

The shadow on the diagonal plane P of a line *ab*, parallel to GL, is vertically projected in $a_2^vb_2^v$ at 45° with GL, found by drawing rays through *a* and *b*, and determining their traces a^v_2, a_2^h, b_2^v, b_2^h on P.

Let (Z, a_2) be the projections of an axis, and the distance of a point *b* in front of that axis be denoted by $\delta = a_2^hx^h$; then the point b_2^v may be found in the following manner: Draw a line through b^v parallel to GL, intersecting Z at x^v; lay off $x^va_2^v = \delta$, and draw $a_2^vc^v$ at 45° parallel to R^h; draw $b^vb_2^v$ at 45°, and their point of meeting b_2^v is the required shadow. If *b* is behind the axis, then δ should be laid off above x^v, instead of below it, as in the figure.

Fig. 33. *Shadow of a circle on the diagonal plane.*

Let the circle be parallel to H, and let it be circumscribed by the square *ad*. Then we have the square $a_2^v d_2^v$ as its projection upon the diagonal plane and any side as $a_2^v b_2^v = ab \cos 45° = ab\sqrt{\frac{1}{2}}$: hence its surface is one-half of that of the original square, and the circle inscribed will be one-half of the original circle.

If the circle had the position indicated in Fig. 34, having the diagonal plane passing through its centre, the vertical projection of the shadow on the diagonal plane is a circle whose radius is the line $\nu_2 = ab \cos 45°$.

Fig. 45. *Application 1st.* Let it be required to find the shadow of the circumference projected in bd on the surface of revolution. Imagine a diagonal plane drawn through the axis of the surface. Then the oblique projection of the circumference bd is obtained as follows: Draw bt at 45°, and zt perpendicular to it; with z as a centre, describe the arc tef: this is the required projection. In the same way the oblique projection of np is xys; they intersect at 1_2 and 2_2, which are the oblique projections of points of the shadow. Drawing lines $1_2 1$ and $2_2 2$ at 45° with GL, we have their intersections 1 and 2 with np as the points required.

Fig. 39. *Application 2d.* Let it be required to find the line of shade of the cone vertically projected in ade. Draw dd_2 at 45°, and od_2 perpendicular to it, with o as a centre and od_2 as a radius, describe a circle; drawing as_2 and ar_2 tangent to this circle, we have the vertical projection of the shadow of the cone on P; drawing the projecting lines $s_2 s$ and $r_2 r$ at 45°, we have ar and as as the required lines of shade of the cone.

Fig. b. *Application 3d. To find the shadow of a given point on the surface of a cone by the method of the diagonal plane.*

Let, Fig. b, Pl. V., abc be a cone, and n the projection of the given point situated at a given distance δ in front of the axis ao. The projection of the cone on the diagonal plane is ab_2c_2; that of the point n is n_2; join a with n_2, and produce it to e_2 on the circumference of shadow; the point e of the base, which casts the shadow e_2, is found by drawing ee_2 at 45°. Now join a with e, and ae is the element upon which the shadow of n is cast at n_1.

Fig. 38. *Application 4th. To find the shadow cast on the surface of a hollow cone by the complete circle of its base.*

Drawing a ray through a, we have x_2 as its oblique projection on the base of the cone; ox_2 is the oblique projection of oa, and ot_2 that of its shadow. Drawing $t_2 t$ at 45°, we have ot the shadow of oa; producing ax_2 to r, we have r as the shadow of the point a. Draw oy_2, oz_2, and oy; then, drawing rays through s and k_2, we have n and k as points of the line of shadow.

Fig. 35 is a diagram used to demonstrate the fact that the line of shade of any surface of the second order is a curve of the same order.

Fig. 40 shows an abridged construction of the shadow of the cylindrical abacus on its column. The highest point b is found by drawing a line mo, making an angle θ with xf; it is the revolved position of a ray in the meridian plane of rays; making the counter revolution, the ray is vertically projected in ob; drawing rb parallel to xf, we have the highest point b. The diagonal plane cuts the column along ed, its line of shade, and the abacus in yk; dk is the oblique projection of the lower circle of the abacus, and d is the point common to this shadow and the line of shade; here the ray becomes tangent to the curve of shadow. The curve of shadow is tangent to sx, and falls between s and r; c is at the same height as a. Any intermediate points, if desired, may be found by the method previously explained.

Fig. 36. As an example of the use of the diagonal plane, let it be required to determine the shadows in Fig. 36.

1st, For *the line of shade of the ovolo*, the construction is the same as that of Fig. 23.

2d, *The shadow of this line* $\alpha\beta$ 45 on the diagonal plane through the axis zs^v, is the curve $\alpha\beta_2 y$. Again, the tangent at β_2 makes an angle of 45° with mn: hence we have the directions of three tangents to the curve; viz., at α, 90°, at β_2, 45°, and at y, 0° with mn.

3d, *The shadow of the edge of the abacus pq on the ovolo.* This line casts a shadow qw on the diagonal plane. (pw is the shadow on V of the edge vertically projected at p.)

We have the highest point, 10, of this shadow by projecting $10'$ on zs^v.

The points 2 and 3 are obtained by taking the intersections 2_2 and 3_2 of the auxiliary shadows wq and $\alpha\beta_2 y$, and projecting back by reversed rays upon the line of shade of the ovolo. These two extreme rays are also tangent to the curve at the points 2 and 3 respectively.

Remark 1. By symmetry, the point 1 is another point of the line of shadow.

Remark 2. The edge of the abacus projected at p, having pw for its shadow, meets the line of shade at 16; hence 16 and 3 are at the same height, for both are symmetrically situated with respect to $4y$.

4th. *The shadow of the abacus on the column.* With w as a centre and a radius equal to zm, describe an indefinite arc 67; it is the indefinite shadow of the edge pq on the column. By Fig. 40 we determine the shadow of $\beta 345$ on the column. $12y$ makes an angle θ, and determines 13; 14 and 15 are at the same height; 8 is determined by the intersection of the line of shade $r8$ with the curve $2_2\beta_2 3_2 y$; the intersection of the two shadows at 7 gives the final shadow on the column, ν, 6, 7, 15, 8.

Remark 3. We have found the shadow by the method of the diagonal plane; but as a verification, and to show the accuracy of the method, we have drawn the plan, and passed a vertical secant plane of rays through any point, as b, taken at random on the edge of the abacus: this plane has $b^h\omega^h$ as its horizontal trace. It cuts from the abacus the line (b^vc, b^h); from the ovolo a curve di (e, o^h); from the fillet (eo, o^h); and from the column $(x\omega^v, \omega^h)$. It is seen that c gives the point c_1, and τ the point 9.

Fig. 37. To find the shades and shadows on the base of a Tuscan column.

The line of shade px of the shaft casts a shadow upon the surface formed by the revolution of bsd called the conge, and also on the torus. To find any point of this shadow, we make use of the diagonal plane. This plane passes through zo and px. Any horizontal plane ts cuts a circumference from the conge, a part of which is projected on the diagonal plane in the arc Vy, intersecting px in y; projecting y on ts by the ray $y\epsilon$, we have ϵ as a point of the shadow of px on the conge; the shadow on the torus can be determined in the same manner. These shadows consist, 1°, of the shadow of the line of shade 1 2 of the fillet: 2°, of that of the arc 1 4; and, 3°, of a portion of py; and are represented by the curve 1_14_15.

Shadows on Sloping Planes.

Fig. 41. The shadow of a, on the plane M passing through GL, and inclined to H at an angle α, is thus found: Denote a_0a^h by δ. From a_0 set off $a_0b = \delta$; draw bn, making the given angle α with GL, and through a^v a right line at 45° with GL: its intersection y with bn is the shadow required.

Fig. 44. To find the shadow of the circle inscribed in the square ad, on a plane passing through GL, and inclined to H at an angle α.

The shadow of the centre o is found at o_1 by taking $o_0c' = o_0o^h$; $c'b_1$ making an angle α with $a'c'$, and o^vs at 45° with GL; $r_1t_1 = ab$; $c_1b_1 = b^vy = ab \cos \alpha$; whence we have the parallelogram $a_1b_1d_1c_1$, and $su = ab \sin \alpha$; the inscribed ellipse is the shadow of the given circle.

Fig. 42. *To find the shadows cast by the chimneys and dormer window upon a sloping roof.*

Let the roof be given as in the figure; $qrp = \alpha$, the angle of the roof with H.

1st, *The shadow of the chimney.* Fig. 43 shows the plan of the chimney. Make $l_0m_3 =$ twice the distance of m in front of l; lay off o_0e and $o_0u = o_0o'$, and o_0o'', Fig. 43, and through y draw a ray yy_1, and through e and u lines parallel to rq, meeting yy_1 in y_1 and y_2; y_1 and y_2 are the shadows of the vertices of the conical chimney-pots, and o_1 and o_2 the centres of the shadows of their upper bases: the shadows of these bases are ellipses found by Fig. 44. Through l_0 and m_3 draw lines parallel to rq; l_0l_1 is the shadow of l_0l; m_1 is the shadow of m; m_1n_1 is the shadow of mn, and n_1o_2 is the shadow of the edge vertically projected at n. Lay off $vx = ms$, Fig. 43, and draw xz_3 parallel to qr; drawing a ray through t, we have t_1 as the shadow of t, and t_1o_2 parallel to ab, as the shadow of the edge us, Fig. 43.

2d, *To find the shadow of the dormer window*, draw $g'l'$; then 1 is situated at a distance 4_01_0 in front of $g'l'$, and also the same distance to the right of 4_0; hence, setting off $4J =$ twice 4_01_0, and drawing 4_04_1 and $J1_1$, and also rays through 1, 2, 3, 4, we have 4_04_1, the shadow of 4_04, and $1\,1_1$ the shadow of the edge vertically projected at 1; in the same way the points 2_1 and 3_1 are determined.

THE HELICOID.

Fig. 49. Let $p =$ the pitch of the helix described by every point of om; α, its angle with the axis.

If we put $\frac{p}{2\pi} = h$ the reduced pitch, we have $c = h \tan \alpha$.

To draw a plane tangent to a warped helicoid at any point a of a given element fk, it is sufficient to draw $At = c$ perpendicular to Ak^h, to join t and a^h; then the perpendicular k^hy dropped from k^h upon ta^h produced, is the horizontal trace of the required tangent plane.

Again, if the trace k^hd^h of the tangent plane containing the element fk be given, to find the point of contact of this plane, it is sufficient to erect the perpendicular $At = c$ to Ak^h, and drop ty upon k^hd^h; ty intersects Ak^h at a^h, the horizontal projection of the required point of contact.

Fig. 46. $Or = m = h \cot \theta$, θ being the angle of the rays with H. The point r is called the *centre of radiation*, and is a fixed point for the same values of h and θ. With o as a centre and $od = c$ as a radius, describe the circumference t^hd: then, with r and the distance c, we can at once find the horizontal projection of the point of shade z^h for any element whatever, as D^h, by erecting a perpendicular od joining d, its intersection with the circumference t^hd, with r; the point z^h, in which dr crosses D^h, is the point required. In this way the curve $1\,0\,3\,2zr\,0^h4$ is obtained.

Fig. 50. Given c, m, the position of r, and the base aek, of a helix, it is required to find that point of the line of shade which touches this helix. From the centre o, with c as a radius, describe a circle $d'd''d'''$. Let r be the centre of radiation, ro being taken equal to m; draw roa; draw ke perpendicular to it; join ad', and with o as a centre and m as a radius, describe an arc rr', meeting ad' produced in r'. Now, if the whole system revolve about a vertical axis at o until r' coin-

cides with the fixed point r, $ad'r'$ will take the direction $1\,dr\,2$, and the point 1 is the horizontal projection of the shade on the element $o1$, making with ok an angle ϕ: 2, situated on *another* element $o2$, is a second point.

Fig. 47 shows this construction with the values of c, m, and ρ, taken from Fig. 46. The triangle $txr = t^vxr''$; $sa = \rho = ok^h$; join a and t, and with s as a centre and a radius $sr = m$, describe the arc ro meeting at produced in o; join s and o, and with a radius $= sl = ok^h$ describe the arc lu; then $lu = k^h1^h$, and determines the point 1.

Fig. 51. The surface here represented is generated by the isosceles triangle $(l^vn^vl_2^v,\ l^hn^h)$ moving around the axis (Z^v, o) so that the vertices describe the three helices lp, ns, and l_2p_2. The direction of the light is denoted by R.

Drawing an indefinite horizontal line ur, Fig. 52, and erecting a perpendicular $sx = \frac{n^vn_2^v}{2}$, and drawing st parallel to l^vn^v, and xr parallel to R′, we have $ts = \pi c$ and $rs = \pi m$; take $sa = \pi \times ol^h$, and draw ato; with s as a centre and a radius sr, draw the arc ro; then with s as a centre and a radius $sl = ol^h$, describe the arc lu. Draw, in Fig. 51, ou parallel to R^h, and from u lay off the arc $u4^h = lu$, Fig. 52, and we have the point 4^h of the line of shade, whence we obtain the vertical projections 1^v, 4^v, on the vertical projection of the helix. The points 2, 3, and 5 are obtained in the same manner. Drawing oX perpendicular to ou, the points 7^h and 8^h will be symmetric with 4^h and 5^h.

Shadows. The shadow of the screw on the coördinate plane H is limited by that of the outer helix, which is determined by points; e.g., the points 4_1, 7_1, and 9_1 are the shadows of 4, 7, and 9.

Shadow cast on the surface by the curve of shade. 1st, To find the shadow cast upon the surface by the curve 4, 5, we use the method of oblique projections (Art. 44); the shadows upon H of the curves 4 5 and 7 8 are $4_1\,5_1$ and $7_1\,8_1$; these, as shown in Fig. 54, intersect in a point y_1; therefore, drawing a ray through y_1, we determine the point y_0, which casts its shadow y upon 7 8. The curves $4\,y_0$ and $y\,7$ have real shadows y_14_1 and y_17_1.

The curve 7 8 is not illuminated. The curve $5\,y_0$ casts the shadow $5^v\,d^v\,y$ on the upper fillet. *To find any intermediate point as* d^v: Draw an element ae; its shadow is $e^hb_1a_1$, intersecting $4_1\,5_1$ at d_1, which, projected back by a reversed ray on to ae, gives the required point d^v, the shadow on the upper fillet of a point of the curve 4 5.

Fig. 48. *Shadow of a helix.* Imagine the helix to be illuminated by rays whose direction is indicated by the arrows; the shadow of any point n is n_1, and the curve of shadow a^hn_1k is a curtate cycloid described by the point a^h, carried around by the circle iqf rolling along the line ii'.

If, in Fig. 52, we take $ai = \rho = ou$, Fig. 51, we have $ij =$ the radius of the rolling circle.

Fig. 53. To show the form (not the position) of the shadow of the outer helix pbl, Fig. 51, we choose at random a centre o, Fig. 53, and with a radius ij, Fig. 52, describe the circle smz; draw ol and os parallel and perpendicular respectively to R^h; draw sg parallel to ol, and ov parallel and equal to yv, Fig. 51; (v is the vertical projection of the horizontal trace of the outer helix pbl, found by producing $p^vb^vl^v$ until it meets GL;) erect at v an indefinite perpendicular; with o as a centre and a radius ou, describe an arc meeting vn in n; n is the initial point of the displaced shadow (its real position being obtained by moving the figure parallel to itself until o coincides with o, Fig. 51); the curve is described by the extremity of the radius vector on when the circumference zms rolls along the line sg.

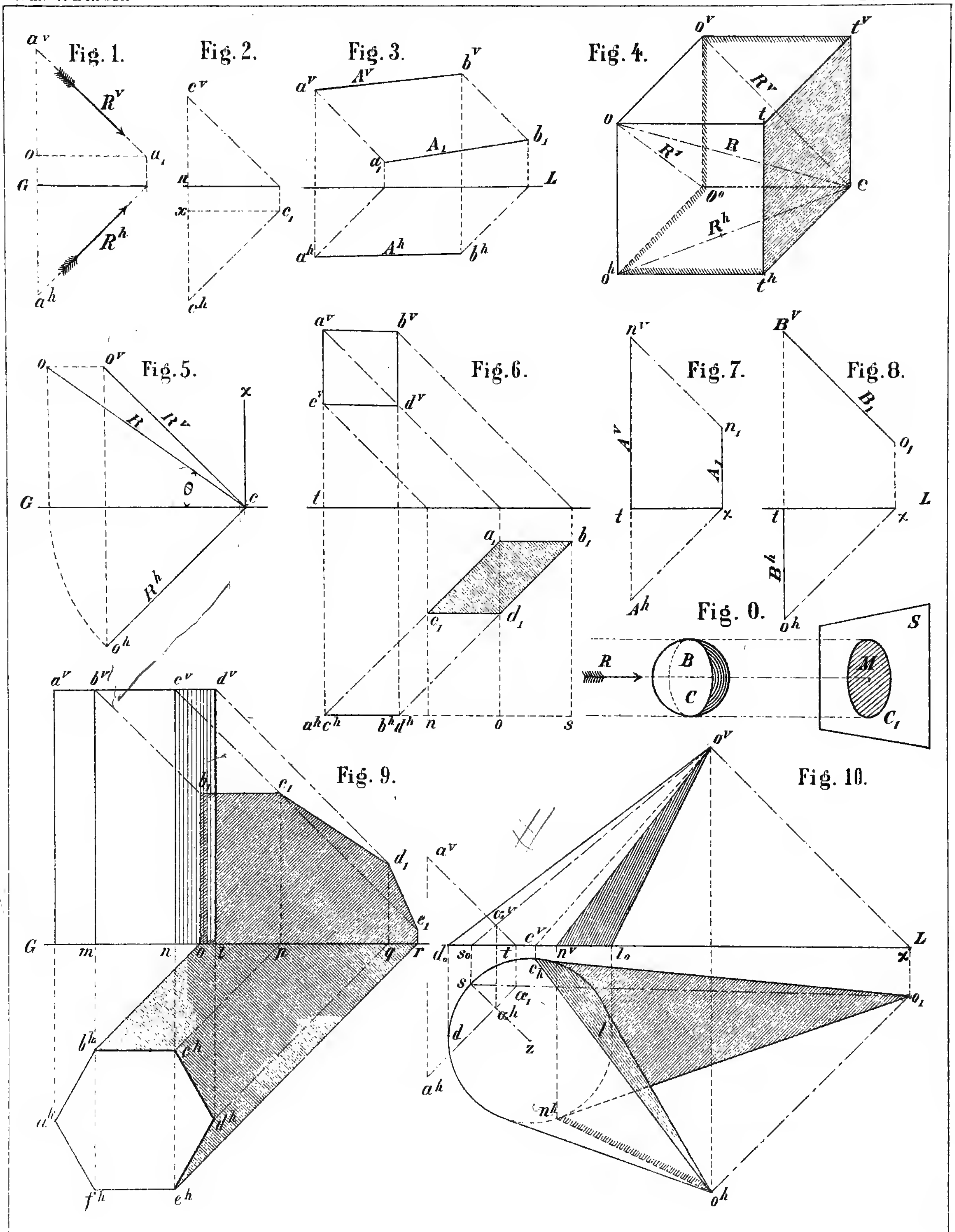
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
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Fig. 6.
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Fig. 8.
Fig. 0.
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Fig. 10.

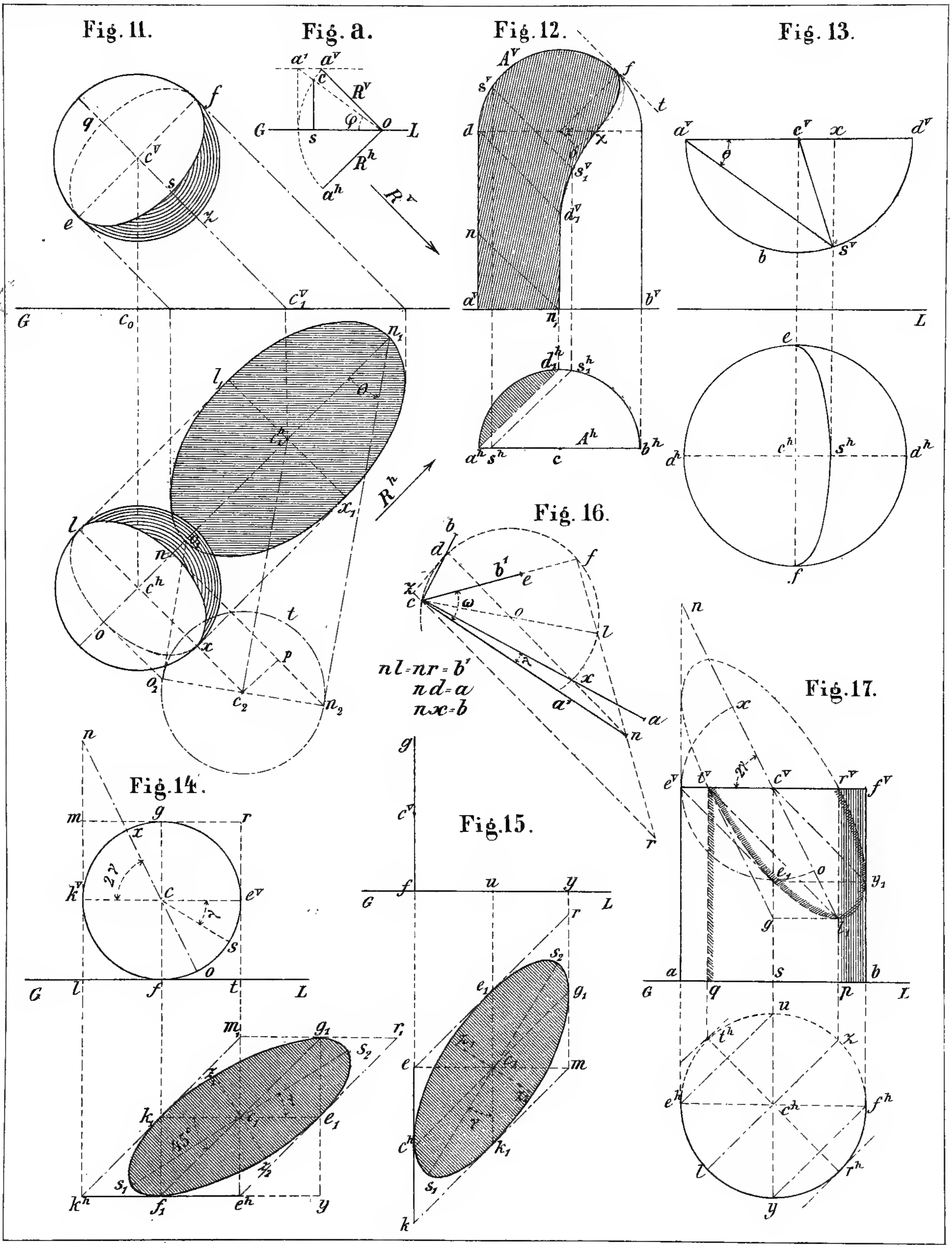
Fig. 11.
Fig. a.
Fig. 12.
Fig. 13.
Fig. 16.
nl = nr = b'
nd = a
nx = b
Fig. 17.
Fig. 14.
Fig. 15.

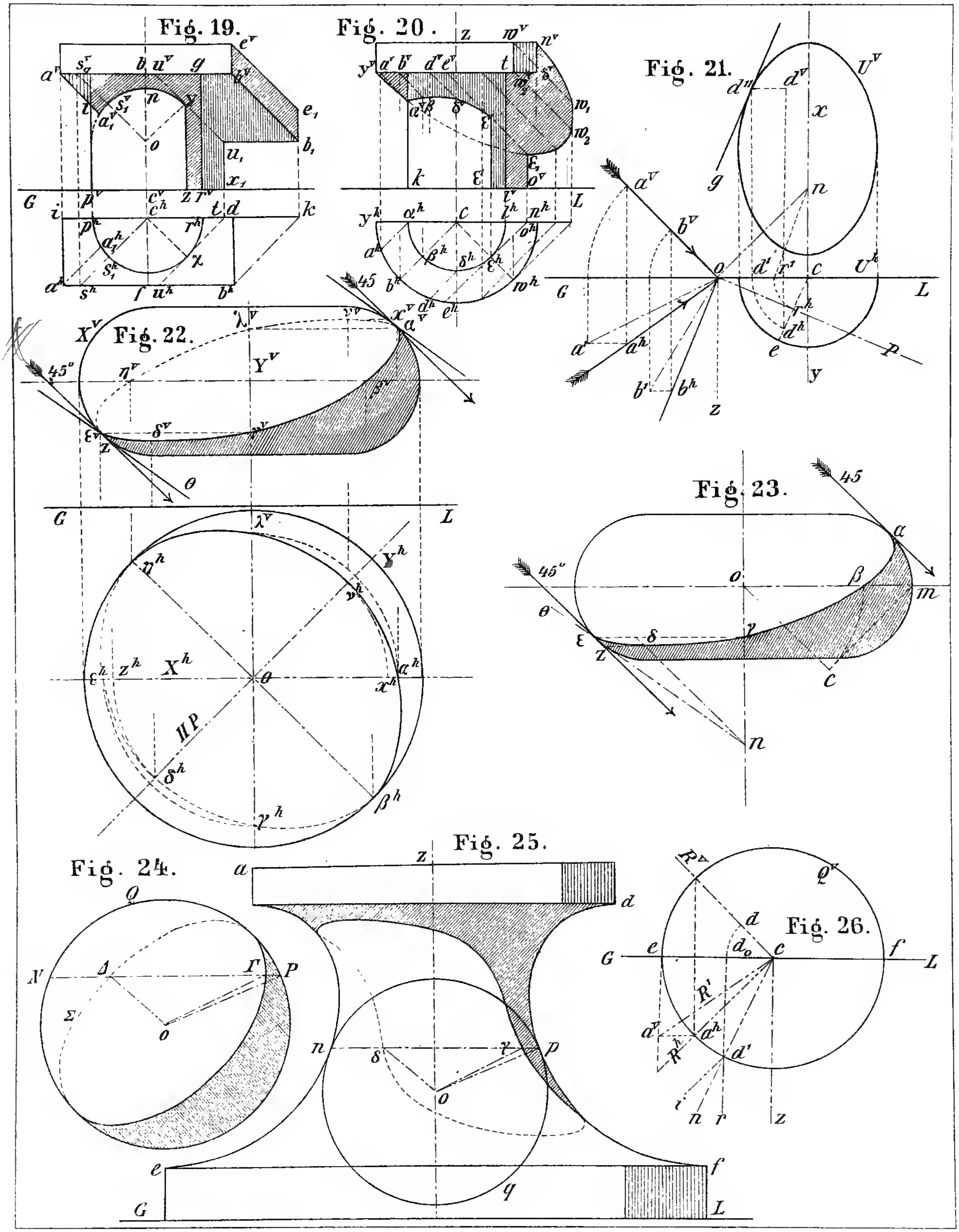

Fig. 19.
Fig. 20.
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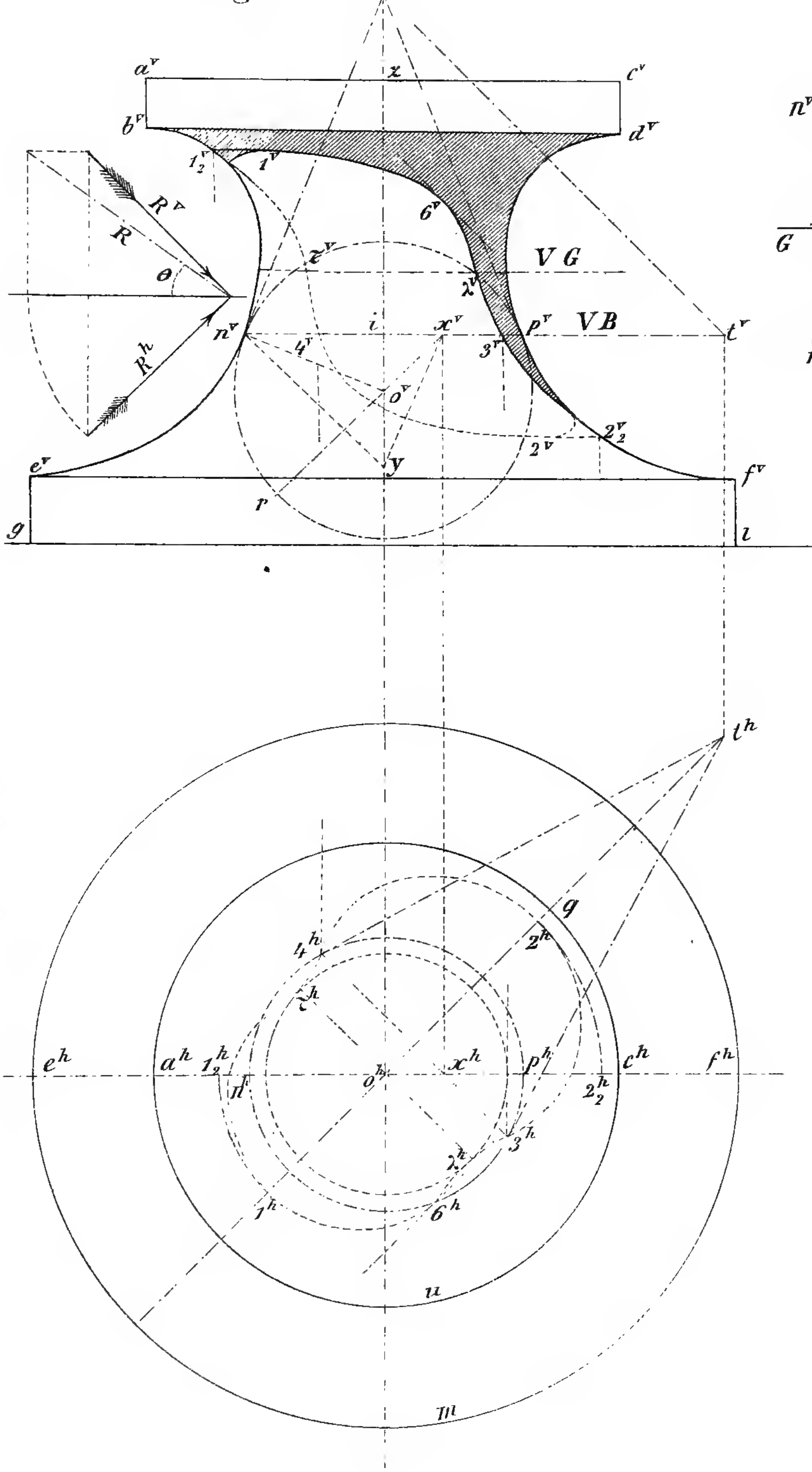
Fig. 27.



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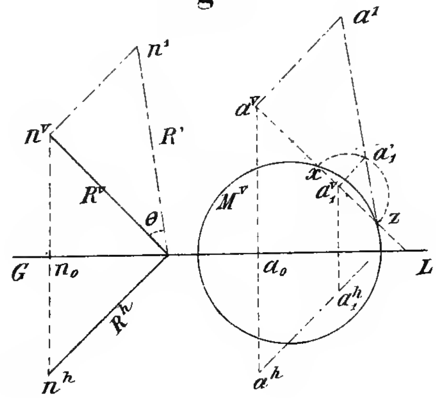
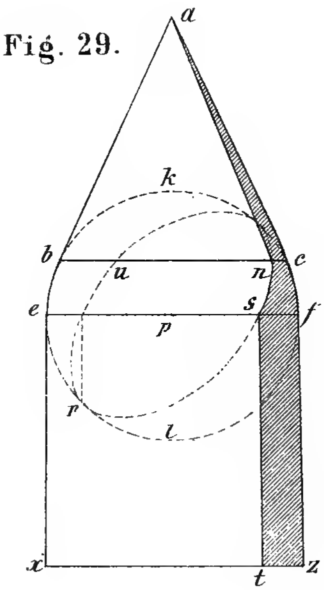


Fig. 29.



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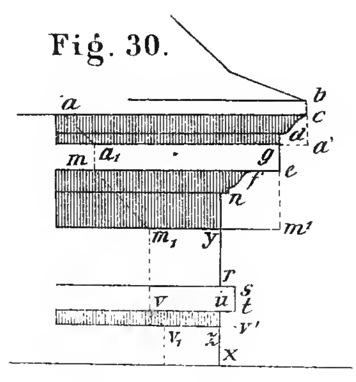


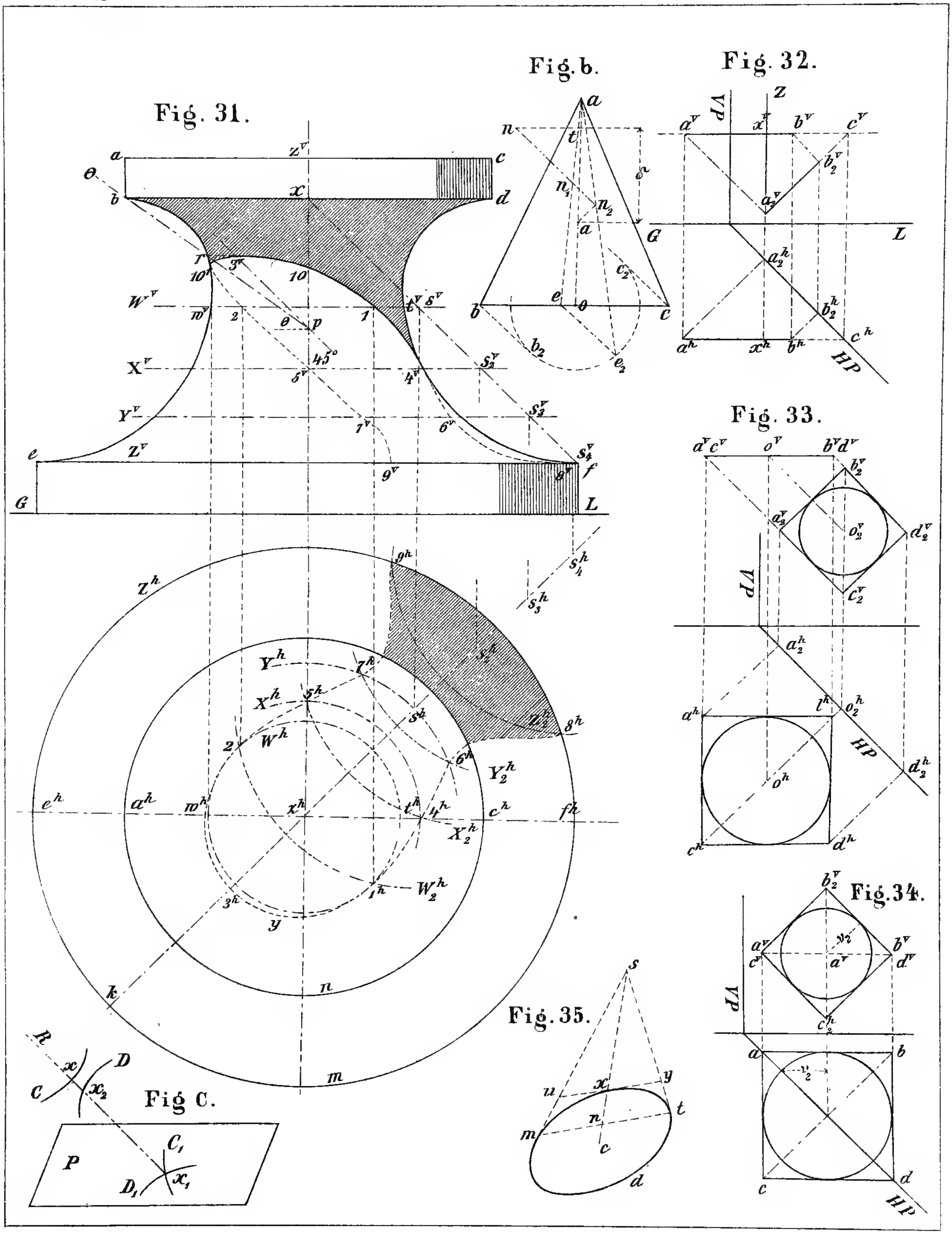
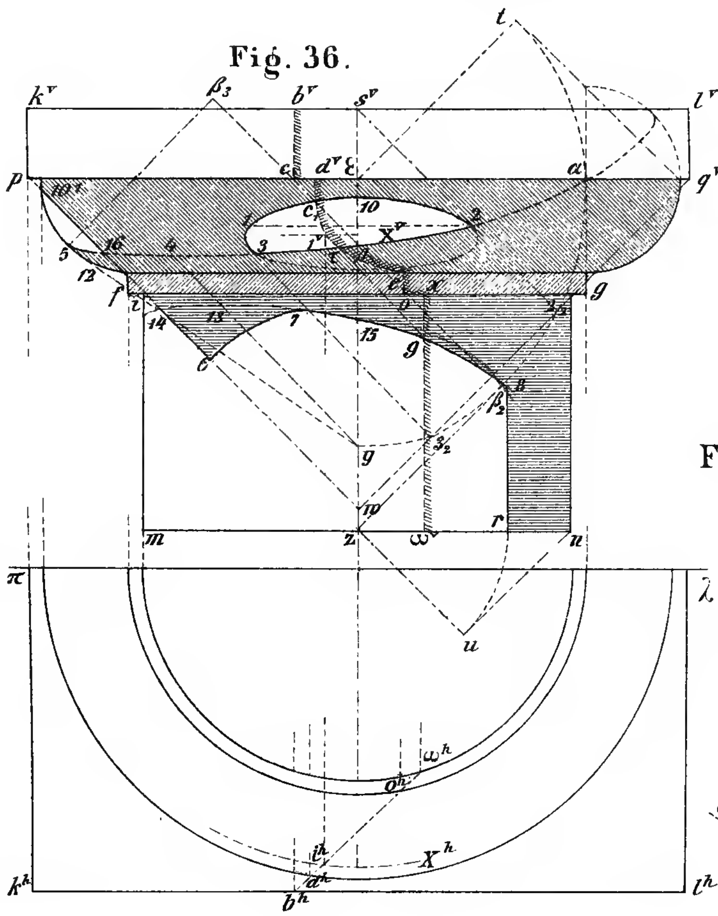


Fig. 31.
Fig. b.
Fig. 32.
Fig. 33.
Fig. 34.
Fig. 35.
Fig C.

Fig. 36.



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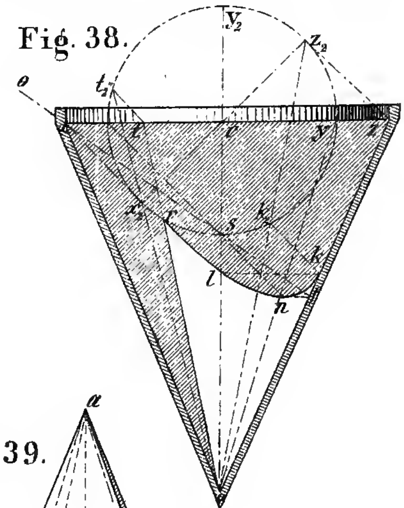

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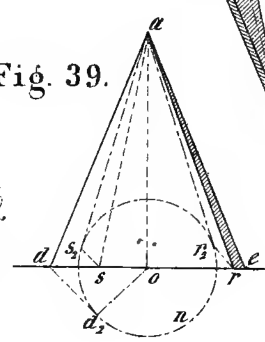

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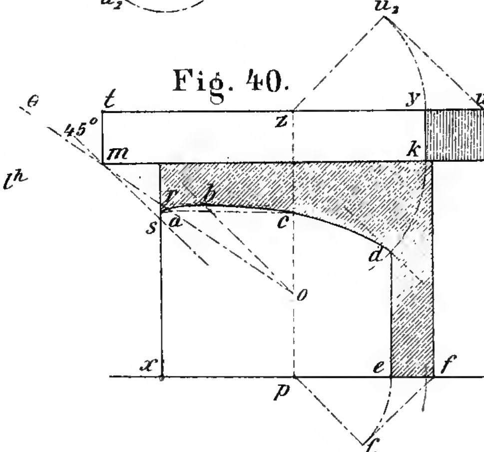

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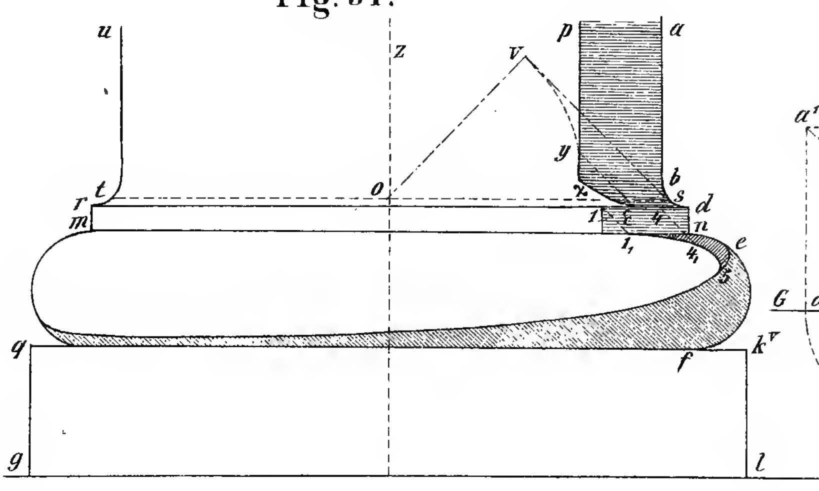


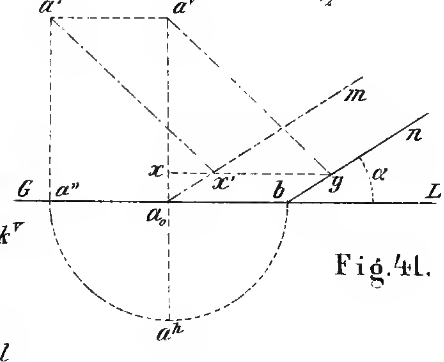


Fig. 41.

Fig. 42.

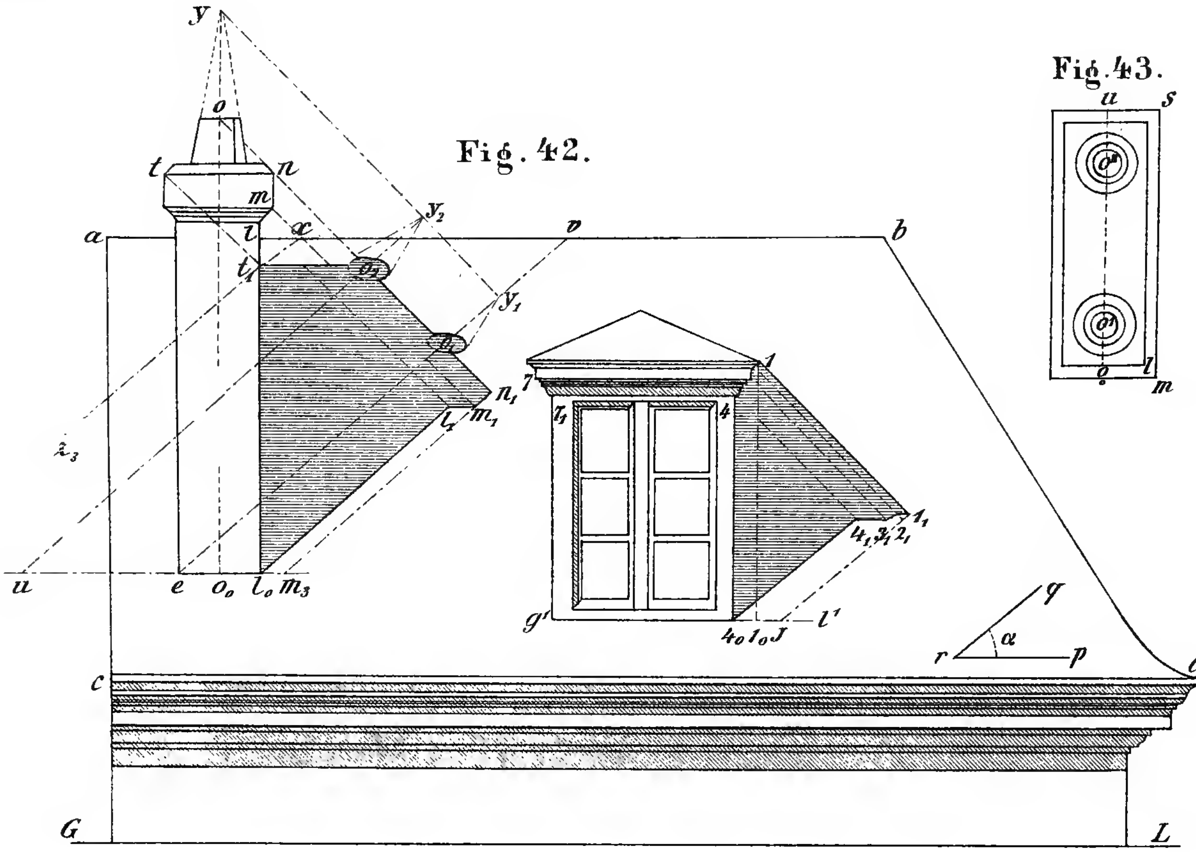
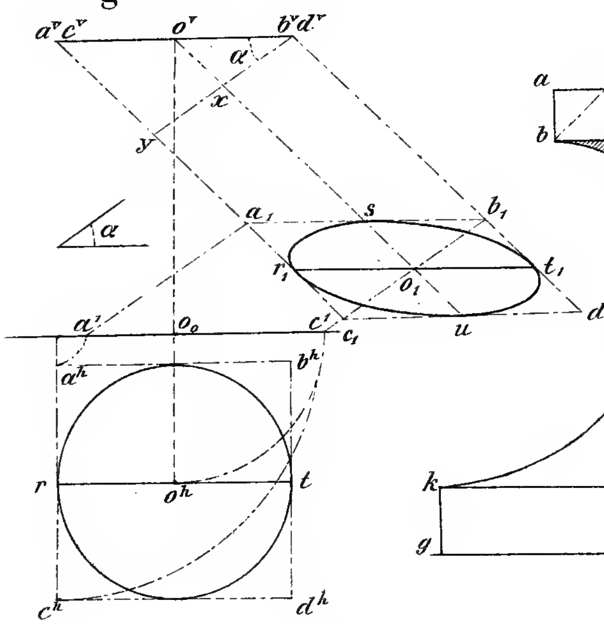


Fig. 43.

Fig. 44.



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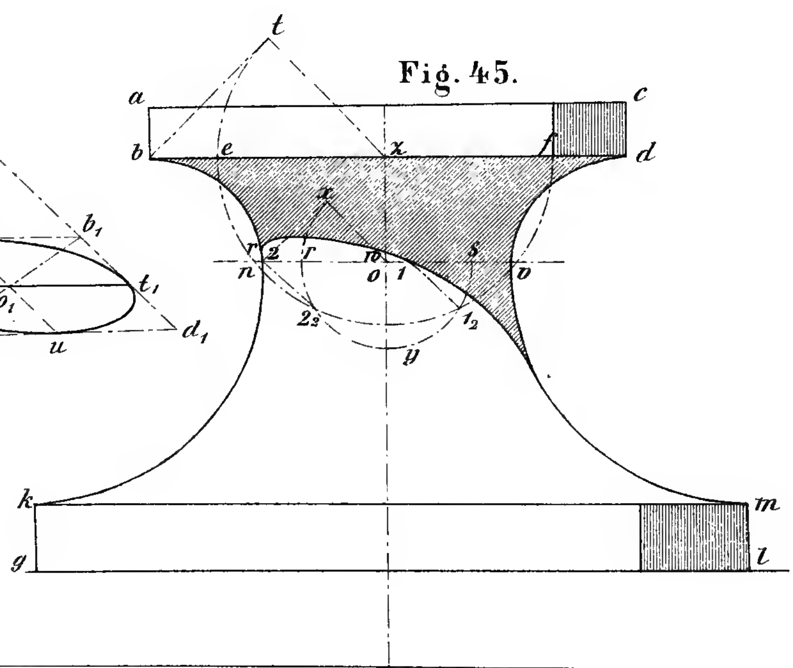

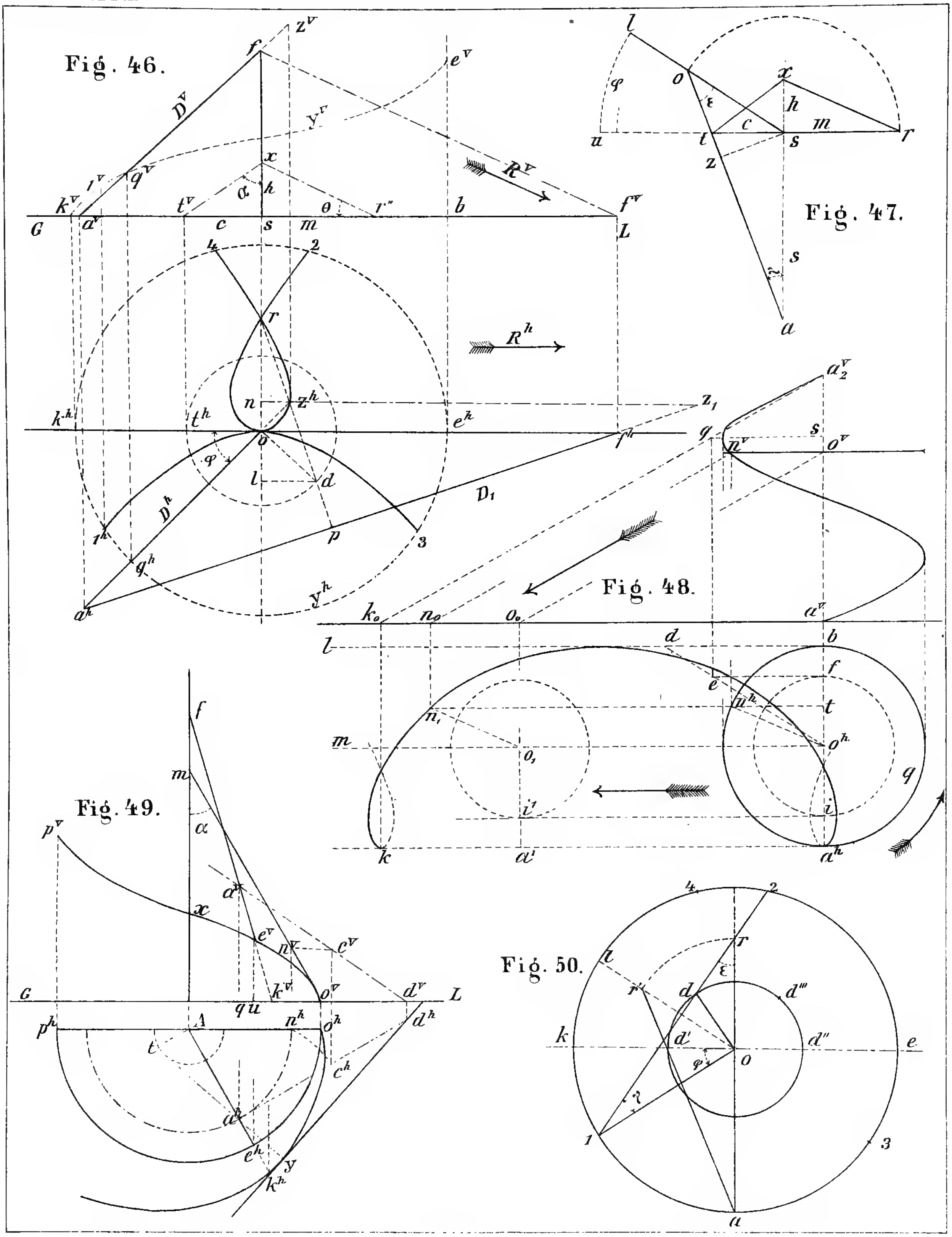
Fig. 46.
Fig. 47.
Fig. 48.
Fig. 49.
Fig. 50.

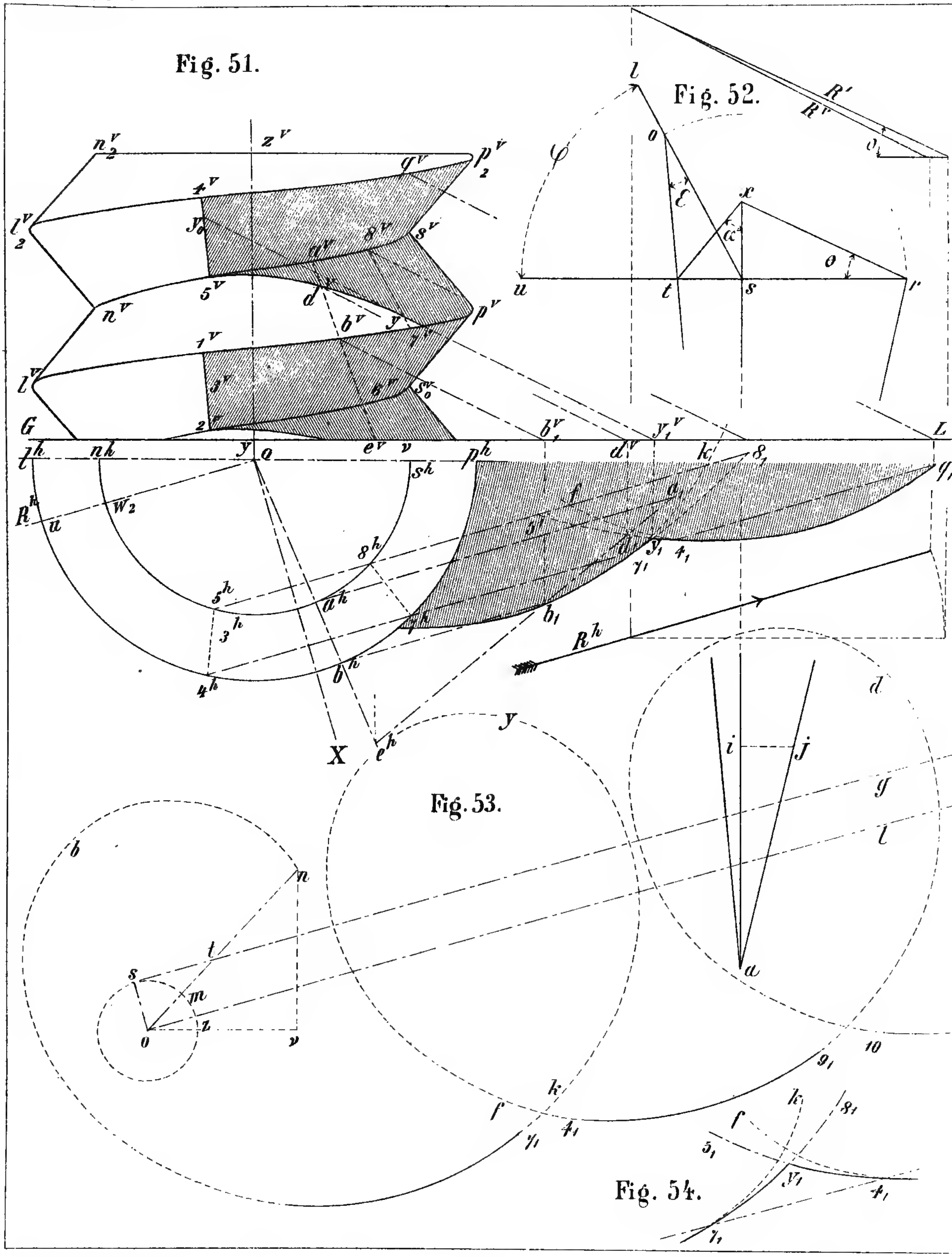

Fig. 51.

Fig. 52.

Fig. 53.

Fig. 54.